Cost Optimization of Step-Down-Stress Accelerated Life Testing

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Abstract. Many products have a high failure rate in their early operating lives. step-down-stress accelerated life testing has been widely accepted as a method of screening out defects before a product is shipped to the customer. In the literature it is often assumed that the failure pattern follows a specific distribution and the step-down-stress accelerated life testing is operated under approximately the same environment as that of the early operating life of the product. In this paper we require the product life distribution to have some specified properties. The step-down-stress accelerated life testing is operated under severe (stress) conditions involving high temperature, voltage, etc. and the product’s residual life depends on the step-down-stress level and the length of step-down-stress accelerated life testing period. Accelerated step-down-stress before shipment will reject poor-quality products and improve product reliability within a warranty period. Accelerated step-down-stress saves time but may cost more. Our goal is to find the appropriate testing parameters to minimize the total of testing, manufacturing, quality and reliability costs. The upper and lower bounds for the optimal step-down-stress accelerated life testing time are derived.

Introduction

Many products have a high failure rate in their early operating lives. Step-down-stress accelerated life testing has been widely accepted as a method of screening out defects before a product is shipped to the customer. A common practice is to test the product until it reaches the change-point where the product failure rate decreases in the infant mortality stage to a constant level in the normal stage [1-3]. [4-7] studied the effect of step-down-stress on the mean residual life of the product. [8-12] studied failure rate model. [13-16] studied economic designs of step-down-stress procedures. [17-19] studied general discussions about step-down-stress.


Step-down-stress accelerated life testing is a new type of life testing in which all the units are subjected to a group of step-down-stress. The stress is kept constant until a fixed time or a fixed number of units fail, and the un-failed units are put into the next experiment with lower stress. The experiment ends up till a fixed time or a fixed number of failure units.

Notations and Assumptions

We assume that every finished product is subject to a step-down-stress accelerated life testing and the product is non-repairable. There is a known relationship between stress conditions and the product
life distribution. Let $\lambda_0$ and $\lambda_i$ be stress level of normal operation and the stress level of step-down-stress test respectively. Their levels may be a function of several stress parameters of operating conditions. Without loss of generality, we assume $\lambda_0 \leq \lambda_i$. The step-down-stress test process may affect the residual life of the product so that the life distribution of the product is not the same as that of no step-down-stress test under normal stress level $\lambda_0$. We assume that the product’s residual life is equivalent to that under a more severe stress level than normal level and that this stress level $\lambda$ is a function of $\lambda_0$ and $\lambda_i$. A typical function form is $\lambda = \left(\frac{\lambda_i}{\lambda_0}\right)^k \lambda_0$, where $0 \leq k \leq 1$.

Let $t_i$ be the step-down-stress time. The life times of the product under stress levels $\lambda_0$, $\lambda_i$ and $\lambda_r$ are denoted by $X_0$, $X_i$ and $X_r$ respectively. The product is scrapped and has a life time $X_i$ if the product fails during the step-down-stress period. The product is shipped to the customer and has a residual life time $X_r$ if the product passes the step-down-stress test. Products with no step-down-stress have life time $X_0$.

**Notations**

These include:

- $c_s$: step-down-stress test set-up cost;
- $c_s(\lambda_i)$: step-down-stress test cost per unit time which is an increasing function of the stress level;
- $c_i$: step-down-stress test failure cost per unit;
- $c_3(t)$: loss of goodwill cost when a failure occurs at time $t$ with the customer;
- $h_i(t)$: hazard rate function under stress level $\lambda_i$, $i = 0, 1$ and $r$;
- $F_i(t)$: distribution function associated with $F_i(t)$.

**Assumptions**

These include:

1. $h_i(t) = \lambda_i g(\lambda_i t)$, $i = 0, 1$ and $r$, where $g(t)$ is a hazard rate function satisfying: $g(t) \to \infty$ as $t \to 0$; $g(t)$ is decreasing for $t \geq 0$; $g(t+s)/g(t)$ is an increasing function of $t$ for $s \geq 0$.

Some often used distributions satisfying assumption (1) are: Weibull distribution with $g(t) = \lambda_i t^{\delta-1} e^{-\lambda_i t}$, $0 < \delta < 1$; Semi-Weibull distribution with $g(t) = \begin{cases} \lambda_i t^{\delta-1} e^{-\lambda_i t}, & 0 \leq t < t^* \\ \delta(t^*)^{\delta-1} e^{-\lambda_i t}, & t \geq t^* \end{cases}$ and Gamma distribution with $g(t) = \Gamma(\alpha) e^{-\lambda_i t} / \int_0^\infty \Gamma(\alpha) e^{-s} ds$, $0 < \alpha < 1$.

2. The loss of goodwill cost $c_3(t)$ and the operating failure cost $c_3(t)$ satisfies: Cost optimization $c_i(t)$ is decreasing for $0 \leq t \leq T_i$ and $c_i(t) = 0$, for $t > T_i$, $i = 2$ and $3$. Here $T_2$ and $T_3$ depend on the warranty period and the target operating life of the product. $c_i(t)$ is differentiable for $0 \leq t \leq T_i$ and $|\partial c_i(t)/\partial t| \leq M$, where $M > 0$.

Some typical function of $c_i(t)$ are constant cost: $c_2 = \begin{cases} c_{2,0} \leq t \leq T_2 \\ 0, & t > T_2 \end{cases}$; Linear decreasing cost: $c_2 = \begin{cases} c_{2,0} \leq t \leq T_2 \\ c_{2,0} - \beta t, & 0, t > T_2 \end{cases}$; and exponentially decreasing cost $c_3(t) = c_3 e^{-\beta e^t}$, $t \geq 0$.

If the product passes the step-down-stress test, then testing for time $t_i$ under stress level $\lambda_i$ is equivalent to using the product for time $t_r$ under stress level $\lambda_r$, where
Here, the residual life is affected by the length of the step-down-stress period as well as the stress level $\lambda$.  

**Cost Model**

If the product fails during the step-down-stress period, the cost per unit consists of the step-down-stress failure cost and the testing cost. The product is shipped to the customer if it passes the step-down-stress test. The cost per unit consists of three parts. They are the step-down-stress cost in the step-down-stress period; the loss of goodwill cost when the product fails in the warranty period; and the operating failure cost which occurs when the product fails during the servicing period. The total cost per unit subject to a step-down-stress time $t_i$ is

$$TC(t_i) = c_2 + c_0(\lambda_1) X_1 I(X_1 \leq t_i) + c_0(\lambda_1) t_i I(X_1 > t_i) + c_2(X_r - t_i) I(X_r > t_r)$$

$$+ c_3(X_r - t_r) I(X_r > t_r), t_r > 0$$

where $I(\cdot)$ is the indicator function. Taking expectation, we have

$$m(t_r) = E[TC(t_i)] = c_2 + c_0(\lambda_1) \int_0^{t_i} s dF_1(s) + c_0(\lambda_1)(1 - F_1(t_i)) + c_2 F_1(t_i)$$

$$+ \int_{t_i}^{t_i + t_f} c_2(s - t_i) dF_1(s) + \int_{t_i}^{t_r} c_3(s - t_i) dF_1(s) = M(t_r)$$

Where

$$t_r = \lambda_1 t_i / \lambda_2$$

from assumption (1) and equation (1).

The total cost for the product without step-down-stress is

$$TC(0) = c_2(X_0) + c_3(X_0),$$

and

$$m(0) = \int_0^{t_f} c_2(s) dF_0(s) + \int_0^{t_r} c_3(s) dF_0(s).$$

**Optimal Step-Down-Stress Time**

The optimal step-down-stress time is determined by minimizing the expected total cost $m(t_r)$ of operating an item in the step-down-stress period and the servicing period. Taking a derivative of the expected total cost with respect to $t_r$,

$$\frac{\partial M(t_r)}{\partial t_r} = (1 - F_1(t_r)) \{c_2(\lambda_1) \lambda_1 / \lambda_2 - (c_2(0) + c_3(0) - c_1) h_1(t_r) + A(t_r) h_r(t_r)\}$$

where

$$A(t_r) = \int_0^{t_r} (-\partial c_2(s)/\partial s) f_1(t_r + s)/f_1(t_r) ds + \int_0^{t_f} (-\partial c_3(s)/\partial s) f_1(t_r + s)/f_r(t_r) ds$$

$$+ c_2(T_2) f_1(t_r + T_2)/f_1(t_r) + c_3(T_3) f_r(t_r + T_3)/f_r(t_r)$$

and $f_r(t)$ is the density function of $F_r(t)$.

**Theorem 1**

(a) If $c_2(0) + c_3(0) \leq c_1$, the optimal step-down-stress time is 0: $m(0) = \min_{t \geq 0}(t_1)$. 

$$F_r(t_r) = F_1(t_1)$$
(b) If \( c_2(0) + c_3(0) > c_1 \), there exists \( t^*_1 \) such that \( m(t^*_1) = \min_{t>0} m(t) \), and \( t^*_1 \) is finite if and only if

\[
c_0(\lambda_1) \frac{\lambda_1}{\lambda_i} - (c_2(0) + c_3(0) - c_1 - A_0) \lambda_1 d > 0,
\]

where \( d = \lim_{t \to -\infty} g(t) \),

\[
A_0 = \lim_{t \to -\infty} A(t) = \int_{0}^{t} (\frac{\partial c_1(s)}{\partial s}) g_s(s) ds + \int_{0}^{t} (\frac{\partial c_2(s)}{\partial s}) g_s(s) ds + c_2(T_2) g_s(T_2) + c_3(T_3) g_s(T_3),
\]

and \( g_s(s) = \lim_{t \to -\infty} f_s(t + s) / f_s(t) \).

The optimal step-down-stress time is greater than 0 when \( m(0) - m(t^*_1) > 0 \). The proof of Theorem 1 is given in the Appendix.

**Theorem 2**

If the optimal step-down-stress time is finite and positive, then

\[
\frac{\lambda_2}{\lambda_i} f^{-1}_r \left[ (\frac{\lambda_2}{\lambda_i}) c_2(0) f_r(T_2) + c_3(0) f_r(T_3) \right] \leq t^*_1 \leq \frac{\lambda_2}{\lambda_i} f^{-1}_r \left[ \frac{c_0(\lambda_i)}{c_2(0) + c_3(0) - c_1} \right]
\]

The proof of Theorem 2 is given in the Appendix.

The results in Theorems 1 and 2 identify the appropriateness of implementing the step-down-stress process before shipment. Theorem 2 gives the upper and lower bounds of the optimal step-down-stress time for a step-down-stress level \( \lambda_1 \). The result is useful in estimating the optimal step-down-stress time numerically.

**Example 1**

Consider a step-down-stress process with the following cost parameters:

- \( g(t) = 0.35t^{0.35-1}, \ t > 0 \).
- \( \lambda_0 = \lambda_r = 0.001 \) ; \( \lambda_i = 0.002 \) ; \( c_s = \$15 \) ; \( c_0(\lambda_i) = 10 \) /hr; \( c_i = \$100 \), \( c_j(t) = \$300 \), for \( 0 \leq t \leq 800 \) ; and 0 otherwise \( c(t) = \$200 \), for \( 0 \leq t \leq 800 \) ; and 0 otherwise.

The optimal step-down-stress time is 1.99 hours and the minimal cost is $281.1 which is $20.6 less than the expected cost of no step-down-stress.

If \( c_j(t) \) is changed to \( c_j(t) = e^{-0.01t^3}, \ for \ 0 \leq t \leq 1200 \) ; and 0 otherwise. Then the optimal step-down-stress time becomes 1.91 hours and the minimal cost is $227.3 which is $30.4 less than expected cost of no step-down-stress.

**Effect of the step-down-stress level.**

It is noted that the step-down-stress cost and the product’s residual life are functions of the step-down-stress level. A high step-down-stress level saves step-down-stress time. However, it may not be economical when the step-down-stress cost is high and a high step-down-stress level causes damage to the product that affects its residual life.

**Example 2**

Suppose that the residual stress level is \( \lambda_r = (\lambda_i / \lambda_0)^k \lambda_0 \) , \( 0.001 \leq \lambda_r \leq 0.01 \) and \( 0 \leq k \leq 1 \). The step-down-stress cost is \( c_0(\lambda_i) = 8 + 2000 \lambda_i + (1000 \lambda_i)^2 \). And other parameters follow Example 1.

It is noted that if the relationship between the step-down-stress level and the residual life distribution is strong, i.e. \( k \) is relatively large, then the optimal step-down-stress level is the normal stress level \( \lambda_0 \). If the residual life distribution after step-down-stress is not seriously affected by the step-down-stress level, i.e. \( k \) is small, then there exists an optimal step-down-stress level \( \lambda^*_1 \). The following theorem gives the result for the case that residual stress level \( \lambda_r \) does not depend on the step-down-stress level \( \lambda_1 \).

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References


