2016 International Conference on Applied Mathematics and Mechanics (ICAMM 2016)
ISBN: 978-1-60595-399-1

Algebraic Topology Approach to Stability Studies of Electrical Power Systems

Leon Luxemburg

Texas A&M University at Galveston, Mathematics, LIST Department, P.O. Box 1675 Galveston Texas, USA 77533

*Corresponding author

Keywords: Differential manifolds, Homology groups, Stability of dynamical systems, Transversality, Electrical power systems, Regions of stability, Boundary of stability region, Morse theory.

Abstract. In this paper we study geometric structure of stability regions of a fairly broad class of dynamical systems, including gradient systems. The lower bounds obtained via an algebraic topology approach can be used to estimate the structure of the boundary of this region and in many cases give the exact number of equilibria corresponding to the bounding surfaces. The results have numerous applications to electrical power systems and to electronic circuits. The methods we use in this study belong in the area of Morse Theory, Algebraic topology and geometric dynamical systems.

Introduction

Consider the following system of differential equations:

$$\begin{aligned} \dot{\delta} &= \omega \\ \dot{\omega} &= \nabla f(\delta) - b\omega \end{aligned} \tag{1}$$

Here f is a twice continuously differentiable function $f: R^n \to R$ which is periodic in each coordinate with some additional generic assumptions given in the following section. This type of system occurs in electrical power systems and analog electronics such as in phase-locked loops and has attracted attention due to its numerous applications to physics and engineering.

The equilibria of this system are the vectors $\delta \in R^n$ for which $f(\delta) = 0$ and the Hessian of this system is the matrix $H(f) = \{\frac{\partial^2 f}{\partial \delta_i \partial \delta_j}\}$ evaluated at $\delta = \{\delta_1, \dots . \delta_n\}$. An equilibrium is called hyperbolic if its Hessian has no eigenvalues on the imaginary axis (in particular, has no zero eigenvalues). The index k of a hyperbolic equilibrium is equal to the number of left half plane complex eigenvalues of the Hessian $H(f)$. An assumption of hyperbolicity of equilibria is known to be generic.

In stability studies of dynamical systems, the distance from a stable equilibrium to the boundary of its region of stability is often used as a measure of robustness of the system. Therefore, it is important to determine the shape of the region of stability which is defined by its bounding surfaces, i.e., stable manifolds of codimension one, which are the stable manifolds of index 1 equilibria lying on the boundary of the stability region. The index $k$of an equilibrium is determined by the number of left half plane eigenvalues of the Hessian $H(f)$.

In some applications, these bounding surfaces are computationally determined [1]. This article establishes lower bounds on the number of index 1 equilibria on the boundary of stability region (i.e. basin of attraction), thus determining a lower bound on the computational load in such studies. Incidentally, it shows that simply using Morse-Smale formulas [3] for the lower bounds for this purpose gives much worse estimates, demonstrating insufficiency of such approach for this case.

Our main result in the next section gives the following lower bounds for the number of bounding surfaces:

$NB(1) \geq 2n(2)$ (2)

where $NB(i)$ is the number of bounding surfaces of codimension 1, which is equal to the number of equilibria of index 1 lying on the boundary of the region of stability. It is clear that these are the most important numbers because the surfaces of higher codimension are of much lower significance in the studies of the boundary of a stability region.

As it turns out, formulas of this type allow in many cases to determine the precise structure of the boundary of stability region, i.e. the exact number of bounding surfaces of each codimension, [6].

See [4]-[10] for many related subjects.

The Main Result

Definition 1. A dynamical system defined on a differential manifold is called Kupka-Smale(KS) is called KupkaSmale if all its equilibria are hyperbolic and the stable manifolds of every equilibrium interests an unstable manifold of any other equilibrium transversally.

In what follows $\beta_i(M, F)$ will denote the i-thBetti number of M, i.e., the rank of i-th homology group of M with coefficients in some field F. In the case of the torus $M = T^n$ it is well known that $\beta_1(M, F) = n$ and this is the only equality we will need for our purposes.

It is well known that KS dynamical systems are generic [2].

Let throughout this article ρ:$\widetilde{M} \to M$ be a universal covering and let *X(ρ)* be the induced vector field on the manifold $\widetilde{M}$ such that $\rho_\#(X(\rho)) = X$, where $\rho_\#$ is the vector bundle map induced by ρ.

Let U be a stable manifold of a stable equilibrium e of X and let $\widetilde{U}$ be some stable manifold of $X(\rho)$ in $\widetilde{M}$ such that $\rho(\widetilde{U}) = U$.

Let $\bar{e}$ be an index 1 equilibrium on the boundary $Bd(U)$ of U in M. For KS vector fields, an equilibrium x lies on the boundary of the stable manifold of an equilibrium y iff there is a trajectory born in x and ending in y, [2]. Therefore, the unstable manifold $U(\bar{e})$ of $\bar{e}$ consists of $\bar{e}$ and two trajectories $x_1(t)$ and $x_2(t)$ of the vector field X such that

$\lim_{t\to-\infty} x_j(t) = \bar{e}$, $\lim_{t\to\infty} x_j(t) = e_j$, j=1,2(3) (3)

where e_j are stable equilibria of X which are not necessarily distinct. We can parameterize each trajectory (together with its end points) by the paths ω_j

$\omega_j : [0,1] \to \{x_j(t): t \in R\} \cup \{\bar{e}\} \cup \{e_j\}$(4) (4)

such that ω_j is a homeomorphism and $\omega_j(0) = \bar{e}, and\ \omega_j(1) = e_j$.

If $e_1 = e_2$ and the loop $\omega_1 \circ \omega_2^{-1}$ is non homotopic to a constant loop in M, then we will call an index 1 equilibrium $\bar{e}$ doubling, otherwise $\bar{e}$ will be called nondoubling. If $e_1 \neq e_2$ then point $\bar{e}$ will be called adjacent (nondoubling), otherwise, nondoubling point is called trivial. From the properties of universal coverings it follows that:

Proposition 1. Let ρ: $\widetilde{U} \to U$ be a restriction of ρ on some stable manifold $\widetilde{U}$ of a stable equilibrium w.r.t.$X(\rho)$, then any doubling point ē on the boundary $Bd(U)$of U has exactly two preimages on $Bd(\widetilde{U})$ and any nondoubling index 1equilibrium (trivial or adjacent) has exactly one preimage on $Bd(\widetilde{U})$.

From (3) it follows that an index 1 equilibrium in $\widetilde{M}$ lies on the boundary of two stability regions (stable manifolds of a stable equilibria) iff $e_1 \neq e_2$, otherwise, it lies on the boundary of just one stability region. It is clear, that a doubling or a trivial point can not lie on the boundary of two stability regions. Clearly,

$E_1(\widetilde{U}) = a(U) + t(U) + 2d(U),$ (5)

where $a(U), t(U), d(U)$ are respectively the numbers of adjacent, trivial and doubling equilibria on the boundary of U and $E_1(\widetilde{U})$ is the total number of index 1 equilibria on the boundary of $\widetilde{U}$. The reason for it is the fact that trivial and adjacent equilibria on the boundary $Bd(U)$ have only one preimage on $Bd(\widetilde{U})$, while doubling points on $Bd(U)$ have exactly two preimages in $Bd(\widetilde{U})$.

Theorem. Assume that the dynamical system given by (1) is KS, and that it has a single stable equilibrium in each period, then its equilibria on the boundary of the stability region satisfy the inequality given by (2).

Proof. First we note that as was shown in [9], see also [8], that under the conditions of this theorem, the number of index 1 equilibria on the boundary of stability region for the system (1) is equal to the number of equilibria on the boundary of stability of the gradient system defined on R^n:$\dot{\delta} = \nabla f(\delta)$. Since function f is periodic, we can factor ∇f through the torus $\nabla f = \nabla g \circ \pi$ where ∇g is a gradient vector field on the torus $M = T^n$ with respect to some function g and $\pi: R^n \to T^n$ is a covering map. Let $M = T^n$ everywhere below and let $M^{(i)}$ be the i-dimensional skeleton resulting from the CW decomposition of M corresponding to its stable manifold CW structure.

From Proposition 1 and (4) it follows that an index 1 equilibrium e_i^1 is doubling if and only if the corresponding inclusion map $f_i: S_i^1 \to M$ is nonhomotopic to zero (where we obtain S_i^1 by identifying the endpoints of the interval [0,1] and we define f_i by the equality $f_i = \omega_1 \circ \omega_2^{-1}$ as in (4). Now, f_i is correctly defined because $\omega_1 \circ \omega_2^{-1}$ is a closed loop). From the assumption that ∇f has only one stable equilibrium on M, it follows that $M^{(1)}$ is a union of unstable manifolds of dimension ≤ 1 then $M^{(1)}$ is a wedge of one dimensional spheres S_i^1, $M^{(1)} = V_{i=1}^k S_i^1$, such that each sphere S_i^1 is a closure of the unstable manifold of index 1 equilibrium e_i^1 and the restriction $h \,/\, M^{(1)}$ of h to $M^{(1)}$ is a homeomorphism onto the 1-skeleton of the CW decomposition of M corresponding to ∇f. Therefore, the inclusion induced homomorphism $i_{\#}: \pi_1(M^{(1)}) \to \pi_1(M)$ is an epimorphism and from (4) it follows that e_i^1 is a doubling point if and only if the corresponding homomorphism $f_{i\#}: \pi_1(S_i^1) \to \pi_1(M)$ induced by f_i is non homotopic to zero. Let l_i be a generator in $\pi_1(S_i^1)$ and let $t_i = f_{i\#}(l_i)$. Consider the following commutative diagram

$$\begin{array}{ccc} \pi_1(M^{(1)}) & \underset{i_{\#}}{\to} & \pi_1(M) \\ H_1 \downarrow & & H_2 \downarrow \\ H_1(M^{(1)}, Z) & \underset{i_*}{\to} & H_1(M, Z) \end{array} \tag{6}$$

where H_1 and H_2 are Hurewicz's epimorphisms. Then, clearly, i_* is also epimorphism. Therefore, $H_1(M, Z)$ is generated by elements $g_i = i_*(H_1(t_i))$. Since F is a field, the Universal Coefficient Theorem implies that $H_1(M, Z) \otimes F = H_1(M, F)$. Therefore, $H_1(M, Z) \otimes F = H_1(M, F)$ is generated by elements $y_i = g_i \otimes 1, i = 1, \ldots, k$. Therefore, the number of elements y_i which are nonzero is at least $\beta_1(M, F) = n$ which implies that the number of elements t_i which are nonzero is also at least $\beta_1(M, F) = n$. The statement of the theorem now follows from the definition of doubling points.

Summary

We have derived an important lower bound which has multiple applications in the electrical power systems, as it allows us to estimate the computational load in stability studies, helps to determine the structure of the boundary of a stability region and provides insight in to the structure of doubling equilibria in engineering applications to dynamical systems. The author has applied these results to the simpler case of phased-locked loops.

Acknowledgement

This research was financially supported by the National Aeronautics and Space Administration.

References

[1] S. Ushiki, "*Analytical expression of unstable manifolds*," Proc. Japan Acad. vol 56, ser A, p. 259, 1980.

[2] J. Palis and W. de Melo, “*Geometric Theory of Dynamical Systems*”, N.Y., Springer Verlag, 1980.

[3] J. Milnor, “*Morse Theory*”, Princeton Univ. Press, Princeton, N.J. 1963.

[4] L. Luxemburg, Cohomology of Decomposition and the Multiplicity Theorem with Applications to Dynamical Systems, Ill. J. Math, Volume 54, Number 2, Summer 2010, Pages 485-499.

[5] L. Luxemburg, Morse-Smale Theory for Flows on Covering Spaces of Compact Manifolds, Journal of Fixed Point Theory and Applications, vol. 9 (2011), no. 2, 197-212.

[6] L. Luxemburg, Sectional Category and its Applications to the Fixed Points of Flows on the Covering Spaces of Compact Manifolds. Topology Proceedings. vol 38, pp 375-398, 2011.

[7] L. Luxemburg, Lower Bounds for Equilibria of Covering Flows on Manifolds, Journal of Dynamical and Control, Systems, Oct. 2010, vol 16, no. 4. pp. 539-555.

[8] L. Luxemburg, Homotopy of dynamical systems on Manifolds and Morse Theory for Covering Spaces, Journal of Dynamical and Control Systems, vol. 16, no 1, pp 59-76, 2010.

[9] L. Luxemburg and G. Huang, Equilibrium Equivalence Theorem and its Applications to Control and Stability Analysis, Circ. Syst. Signal Proc., 1995, vol. 14, No 1, pp 111-134.

[10] L. Luxemburg and G. Huang, Generalized Morse Theory and Applications to Control and Stability Analysis, Circ. Syst. Signal Proc., 1991, vol. 10, No 2, pp. 175-209.