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The Vertex-Distinguishing Edge Coloring of $P_m \vee K_n$ and $C_m \vee K_n$

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Abstract. A proper edge coloring of graph G is called equitable adjacent strong edge coloring if colored sets from every two adjacent vertices incident edge are different, and the number of edges in any two color classes differ by at most one, which the required minimum number of colors are called the adjacent strong equitable edge chromatic number. In this paper, we obtain vertex-distinguishing edge coloring of $P_m \vee K_n$ and $C_m \vee K_n$.

Introduction

The problem about vertex-distinguishing edge coloring of G is a widely used and extremely difficult problem [1-4]. In [5] introduced the vertex-distinguishing edge coloring of graph, and give the correspondence conjecture.

Definition 1[6] G is a simple graph and k is a positive integer, if it exists a mapping f , $E(G) \xrightarrow{f} \{1,2,\cdots,k\}$, and satisfied with $f(e) \neq f(e')$ for adjacent edge $e, e' \in E(G)$, then f is called a proper edge coloring of G, is abbreviated $k-PEC$ of G, and is called the *Edge Chromatic Number of* G.

$$\chi'(G) = min\{k|\ k-PEC\ of\ G\}$$

Definition 2[1-4] For the proper edge coloring f of simple graph, if it is satisfied with $C(u) \neq C(v)$ for $V(G)(u \neq v)$, where $C(u) = \{f(uv)|\ uv \in E(G)\}$, then f is called the Vertex-distinguishing Edge Coloring, is abbreviated $k-VDEC$ of G, and is called the *Vertex-distinguishing Edge Chromatic Number of* G.

$$\chi'_{vd}(G) = min\{k|\ k-VDEC\ of\ G\}$$

Definition 3 For a graph G, n_i is the vertex number which degree is i, using δ, Δ denoted the minimum, maximum degree of G, it is called

$$\mu(G) = max\left\{min\left\{\lambda|\binom{\lambda}{2} \geq n_i\right\}, \delta \leq i \leq \Delta\right\}$$

Combinatorial Degree of G.

Conjecture For a connected graph G and $|V(G)| \geq 3$, then

$$\mu(G) \leq \chi'_{vd}(G) \leq \mu(G) + 1,$$

The left of the conjecture is obviously true.

Definition 4[6] Suppose G and H are two simple graphs which are vertex disjointed and edge disjointed,

$$V(G \vee H) = V(G) \cup V(H),$$
$$E(G \vee H) = E(G) \cup E(H) \cup \{uv|u \in V(G), v \in V(H)\},$$

Then $G \vee H$ is called a Join-graph of G and H.

$P_1 \vee K_n = K_{n+1}, P_2 \vee K_n = K_{n+2}$ has been discussed In [5], and $P_m \vee K_3 = P_m \vee C_3$ has also been discussed in another paper. Let $m \geq 3, n \geq 4$ in this paper.

Lemma 1 If $m \geq 3, n \geq 4$

$$\mu(P_m \vee K_n) = m + n .$$

Proof For $m = 3, n = 5$,

$$max\left\{min\{\theta|\tbinom{\theta}{6} \geq 2\}, min\{\theta|\tbinom{\theta}{7} \geq 6\}\right\} = 8.$$

For $m \geq 3, n \geq 4, m + n \neq 8$,

$$max\left\{min\{\theta|\tbinom{\theta}{n+1} \geq 2\}, min\{\theta|\tbinom{\theta}{n+2} \geq m-2\}, min\{\theta|\tbinom{\theta}{m+n-1} \geq n\}\right\} = m + n.$$

So lemma 1 is true.

Lemma 2[5] For a complete graph K_n,

$$\chi'_{vd}(K_n) = \begin{cases} n-1, for\ n \equiv 0(mod\ 2) \\ n, \quad for\ \ n \equiv 1(mod\ 2) \end{cases}$$

Lemma 3 If $m \geq 4, n \geq 4$,

$$\mu(C_m \vee K_n) = m + n .$$

Proof $\mu(C_m \vee K_n) = max\left\{min\{\theta|\tbinom{\theta}{n+2} \geq m\}, min\{\theta|\tbinom{\theta}{m+n-1} \geq n\}\right\} = m + n.$

The terms and signs we use in this paper but not denoted can be found in [6-8].

Adjacent Strong Edge Coloring of $P_m \vee K_n$

Theorem 2.1 If $m + n \geq 3$,

$$\chi'_{vd}(P_m \vee K_n) = \begin{cases} n+1, m = 1\ and\ n \equiv 0(mod\ 2) \\ n+2, m = 1\ and\ n \equiv 1(mod\ 2)\ \ or\ m = 2\ and\ n \equiv 1(mod\ 2) \\ n+3, m = 1\ and\ n \equiv 0(mod\ 2) \end{cases}$$

Proof When $m = 1,2$, we can get $P_m \vee K_n = K_{m+n}$ from [5], the conclusion is true.

Theorem 2.2 If $m \geq 3, n \geq 4$

$$\chi'_{vd}(P_m \vee K_n) = m + n .$$

Proof Denote $P_m = u_1 u_2 \cdots u_m, V(K_n) = \{v_i | i = 1,2,\cdots,n\}$

$$C = \{1,2,\cdots,m+n-1\}, \bar{C}(u) = C \backslash C(u), u_i = v_{n+i}, i = 1,2,\cdots,m.$$

From lemma 2, we need prove $P_m \vee K_n$ is $(m+n) - VDEC$ only.

Suppose f is

$$f(v_i v_j) = i + j - 2(mod\ (m+n)), i = 1,2,\cdots n; j = i+1; i+2,\cdots,m+n .$$

Case 1 If $m > n \geq 4$

$$f(u_i u_{i+1}) = i, i = 1,2,\cdots,m-1.$$

Then

$\bar{C}(v_i) = \{2(i-1)\}, i = 1,2,\cdots,n$;

$C(u_1) = \{1; n, n+1,\cdots,2n-1\}$;

$C(u_m) = \{m-1; m+n-1,0,1,\cdots,n-2\}$;

$C(u_i) = \{i-1, i; n+i-1,\cdots,2n+i-2\}(mod(m+n)), i = 2,3,\cdots,m-1.$

Thus f is $(m+n) - VDEC$ of $P_m \vee K_n$. This proves the result is true.

Case 2 If $m = n$,

$$f(u_i u_{i+1}) = i, i = 1,2,\cdots,n-1.$$

Then

$\bar{C}(v_i) = \{2(i-1)\}, i = 1,2,\cdots,n$;

$C(u_1) = \{1; n, n+1,\cdots,2n-1\}$;

$C(u_m) = \{n-1; 2n-1,0,1,\cdots,n-2\}$;

$C(u_i) = \{i-1, i; n+i-1, n+1,\cdots,2n+i-2\}\big(mod(2n)\big), i = 2,3,\cdots,n-1$;.

Thus f is $(m+n)-VDEC$ of $P_m \vee K_n$. This proves the result is true.

Case 3 If $n > m$,

$f(u_i u_{i+1}) = n-m+i, i = 1,2,\cdots,m-1.$

Then

$\bar{C}(v_i) = \{2(i-1)\}, i = 1,2,\cdots,\frac{m+n}{2}$;

$\bar{C}(v_i) = \{2i-m-n+1)\}, i = \frac{m+n}{2}+1, \frac{m+n}{2}+2,\cdots,n$;

For $m+n \equiv 1(mod\ 2)$

$\bar{C}(v_i) = \{2(i-1)\}, i = 1,2,\cdots,\frac{m+n+1}{2}$;

$\bar{C}(v_i) = \{2i-m-n)\}, i = \frac{m+n+1}{2}+1, \frac{m+n+1}{2}+2,\cdots,n$;

$C(u_1) = \{n-m+1, n, n+1,\cdots,2n-1\}(mod(m+n))$;

$C(u_m) = \{n-1; m+n-1,0,1,\cdots,n-2\}$;

$C(u_i) = \{n-m+i, n-m+i+1; n+i-1, n+i,\cdots,2n+i-2\}\big(mod(m+n)\big), i =$
$2,3,\cdots,m-1.$

Thus f is $(m+n)-VDEC$ of $P_m \vee K_n$. This proves the result is true.

All in all, we conclude that theorem 2.2 is true.

Adjacent Strong Edge Coloring of $\mathbf{C_m} \vee \mathbf{K_n}$

Theorem 3.1 If $n \geq 1$,

$$\chi'_{vd}(C_3 \vee K_n) = \begin{cases} n+4, for\ n \equiv 1(mod\ 2) \\ n+3, for\ n \equiv 0(mod\ 2) \end{cases}.$$

Proof Because of $C_3 \vee K_n = K_{n+3}$, the result is true we know by [5].

Theorem 3.2 If $m \geq 4, n \geq 4$,

$$\chi'_{vd}(C_m \vee K_n) = m+n.$$

Proof Obviously $\chi'_{vd}(C_m \vee K_n) \geq \mu(C_m \vee K_n)$. By lemma 2, we need prove $C_m \vee K_n$ has a mapping $(m+n)-VDEC$ only. Denote

$$C_m = u_1 u_2 \cdots u_m u_1, V(K_n) = \{v_i | i = 1,2,\cdots,n\},$$
$$C = \{1,2,\cdots,m+n-1\}, \bar{C}(v) = C \backslash C(v), u_i = v_{n+i}, i = 1,2,\cdots,m.,$$

Case 1 If $m > n$,

Suppose f is

$f\big(v_i v_j\big) = i+j-2\big(mod\ (m+n)\big), i = 1,2,\cdots n; j = i+1; i+2,\cdots,m+n$;

$f(u_iu_{i+1}) = i, i = 1,2,\cdots,m-1$;
$f(u_mu_1) = n-1$.

Then

$\bar{C}(v_i) = \{2(i-1)\}, i = 1,2,\cdots,n$;
$C(u_1) = \{1, n-1, n, \cdots, 2n-1\}$;
$C(u_m) = \{m-1, m+n-1, 0, 1, \cdots, n-1\}$;
$C(u_i) = \{i-1, i, n+i-1, n+i, \cdots, 2n+i-2\}\big(mod(m+n)\big), i = 2,3,\cdots,m-1$.

Thus f is $(m+n)-VDEC$ of $C_m \vee K_n$. This proves the result is true.

Case 2 If $m = n$,

Suppose f is

$f(v_iv_j) = i+j-2\big(mod\ (2n)\big), i = 1,2,\cdots n; j = i+1; i+2,\cdots,2n$;
$f(u_iu_{i+1}) = i+1, i = 1,2,\cdots,n-1$;
$f(u_nu_1) = n-1$.

Then

$\bar{C}(v_i) = \{2(i-1)\}, i = 1,2,\cdots,n$;
$C(u_1) = \{2, n-1, n, \cdots, 2n-1\}$;
$C(u_m) = \{n-1, n, 2n-1, 0, 1, \cdots, n-2\}$;
$C(u_i) = \{i, i+1, n+i-1, n+i, \cdots, 2n+i-2\}\big(mod(2n)\big), i = 2,3,\cdots,n-1$.

Thus f is $2n-VDEC$ of $C_m \vee K_n$**.** This proves the result is true.

Case 3 If $m > n$,

Suppose f is

$f(v_iv_j) = i+j-2\big(mod\ (m+n)\big), i = 1,2,\cdots n; j = i+1; i+2,\cdots,m+n$ *;*
$f(u_iu_{i+1}) = n-m+i, i = 1,2,\cdots,m-2$;
$f(u_{m-1}u_m) = n$;
$f(u_mu_1) = n-1$.

Then

If $m+n \equiv 0(mod\ 2)$,

$\bar{C}(v_i) = \{2(i-1)\}, i = 1,2,\cdots,\frac{m+n}{2}$;

$\bar{C}(v_i) = \{2i-(m+n)-1\}, i = \frac{m+n}{2}+1, \frac{m+n}{2}+2, \cdots, n$;

$C(u_1) = \{n-m+1, n-1, n, \cdots, m+n-1\}$;

$C(u_{m-1}) = \{n-2, n, m+n-2, m+n-1, 0, 1, \cdots, n-3\}$;

$C(u_m) = \{n-1, n, m+n-1, 0, 1, \cdots, n-2\}$;

$C(u_i) = \{n-m+i, n-m+i+1, n+i-1, n+i, \cdots, n-m+i-2\}\big(mod(m+n)\big), i =$ $2,3,\cdots,m-2$.

If $m+n \equiv 1(mod\ 2)$,

$\bar{C}(v_i) = \{2(i-1)\}, i = 1,2,\cdots,\frac{m+n+1}{2}$;

$\bar{C}(v_i) = \{2(i-1)-m-n\}, i = \frac{m+n+1}{2}+1, \frac{m+n+1}{2}+2, \cdots, n$;

$C(u_1) = \{n-1, n-m+1, n, n+1, \cdots, n+m-1\}$;

$C(u_{m-1}) = \{n-2, n, m+n-2, m+n-1, 0, 1, \cdots, n-3\}$;

$C(u_m) = \{n-1, n, m+n-1, 0, 1, \cdots, n-2\}$;

$C(u_i) = \{n-m+i-1, n-m+i, n+i-1, n+i, \cdots, n-m+i-2\}\big(mod(m+n)\big), i = 2, 3, \cdots, m-2$.

Thus f is $(m+n)-VDEC$ of $C_m \vee K_n$. This proves the result is true.
All in all, we conclude that theorem 3.2 is true.

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