A Structure of Some Abundant Semigroups

Deju Zhang¹,a, Xiaomin Zhang²,b

¹School of Science, Linyi University, Linyi, Shandong, 276005, P.R. China
²School of Logistics, Linyi University, Linyi, Shandong, 276005, P.R. China

azhangdeju@lyu.edu.cn, bygxxm1992@126.com

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Abstract. Semigroup theory is widely applied in many disciplines such as computer science, operation theory and combinatoric mathematics etc. In this paper, we introduce the concept of perfect ♯-abundant semigroups of type C and we overcome the difficulty about results on ♯-abundant which is a little than abundant semigroups, A construction theorem for such semigroups will be given.

1 Introduction

Semigroup theory was founded in early 1930s, it has become a unique discipline in algebra. Semigroup theory has widely applied in many disciplines such as computer science, operation theory, combinatoric mathematics. In recent years, with Cross and integration of semigroup theory, form Language, automaton theory and cryptography, it get rapidly development. Generalized regular semigroups in semigroup theory has become research hotspots.

Semigroup $S$ is called rpp if for any $a \in S$, $aS^1$ regarded as an $S^1$-system is projective. Dually, we can define lpp semigroups. In order to investigate well rpp semigroups, Fountain [1] and [2] introduced usual Green's *-relation $L, R, H, D$ and $J$. For elements $a, b$ of a semigroup $S$, define $aLb$ if and only if for all $x, y \in S^1, ax = ay \Leftrightarrow bs = by$. The relation $R$ on $S$ can be defined Dually. It is obvious that $L$ is a right congruence and $R$ is a left congruence. He pointed out that a semigroup $S$ is rpp if and only if each $L$ -class of $S$ contains at least an idempotent. $S$ is called abundant if each $L$ -class and each $R$ -class of $S$ contains at least an idempotent. Equivalently, a semigroup $S$ is abundant if and only if it is both rpp and lpp. Abundant semigroups were first investigated by Fountain [2]. Since then, abundant semigroups have been extensively studied by El-Qallali, Fountain, Guo and others (see[1-10]). Furthermore, Fountain [1] called an abundant semigroup $S$ adequate if the idempotent of $S$ form a semilattice. Indeed, the adequate semigroups are the analogue of the inverse semigroups with the class of abundant semigroups. As an analogue of orthodox semigroups in the class of abundant semigroups, El-Qallali [4] called an abundant semigroups $S$ a quasi-adequate semigroup if the idempotents $S$ a subsemigroup of $S$.

In a recent paper[7], Guo-Shum have introduced the concept of perfect abundant semigroups, they proved that a semigroup $S$ is perfect abundant if and only if $S$ is strongly rpp and strongly lpp. some results on perfect rpp semigroups are strengthened similar to perfect rpp semiroups, Peng-Kong[11] introduced the perfect right ♯-abundant semigroups, and obtained the structure theorem of such a semi-group. In this paper, we introduce the concept of perfect ♯-abundant semigroups of type C and we overcome the difficulty about results on ♯-abundant which is a little than abundant semigroups, A construction theorem for such semigroups will be given by means of the methods of [7].

For terminology and notations not give in this paper, the reader is referred to Howie[9] and Petrich[11].
2 Preliminaries

We first recall that the Green's $\gamma$- relations defined in [10].

$$\mathcal{L}^2 = \{(a,b) \in S | \forall (x,y) \in E(S^1) \text{ and } ae = af \iff be = bf\},$$

$$\mathcal{R}^2 = \{(a,b) \in S | \forall (x,y) \in E(S^1) \text{ and } ea = fa \iff eb = fb\},$$

$$\mathcal{H}^2 = \mathcal{L}^2 \cap \mathcal{R}^2, \mathcal{D}^2 = \mathcal{L}^2 \cup \mathcal{R}^2.$$

We easily check that the relations $\mathcal{L}^1$ and $\mathcal{R}^2$ are equivalent relation. However, $\mathcal{L}^2$ is not a right compatible (that is, right congruence), $\mathcal{R}^2$ is not a left compatible (that is, left congruence), and $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \mathcal{L}^2, \mathcal{R} \subseteq \mathcal{R}^* \subseteq \mathcal{R}^2$. A semigroup $S$ is right $\gamma$-abundant if each $\mathcal{L}^2$-class of $S$ contains at least one idempotent. We can define left $\gamma$-abundant semigroups dually. A semigroup $S$ is called $\gamma$-abundant if it is both right $\gamma$-abundant and left $\gamma$-abundant. A $\gamma$-abundant semigroup $S$ is called $\gamma$-quasi-adequate if $E(S)$ forms a subsemigroup of $S$. A $\gamma$-quasi-adequate semigroup is called $\gamma$-adequate if $E(S)$ forms a semilattice. Abundant semigroups is proper subclass of $\gamma$-abundant semigroups (see[10]), and if $a, b$ are regular elements of $S$, then $a\mathcal{L}^2b \iff a\mathcal{L}^2b (\text{or } a\mathcal{L}^2b)(\text{see}[10]).$ A $\gamma$-abundant(quotient-adequate, adequate) semigroup is called type C if $S$ is a right congruence and $\mathcal{R}^2$ is a left congruence on $S$.

Definition 2.1 A $\gamma$-right abundant semigroup is called strongly $\gamma$-right abundant if every $a \in S$, there exists a unique idempotent $e \in \mathcal{L}^2_e \cap E(S)$ such that $ea = a$. A strongly $\gamma$-right abundant $S$ is called perfect $\gamma$-right abundant if $\mathcal{L}^2$ is a congruence on $S$ and $E(S)$ is a normal band.

Lemma 2.2 Let $S$ be a $\gamma$-right abundant semigroup, and $e \in E(S)$. Then the following are equivalent:

(i) $(e, a) \in \mathcal{L}^2$;

(ii) $a = ae$ and $ag = ah \iff eg = eh$ for all $g, h \in E(S^1)$.

$\mathcal{L}^2$-class of $S$ contains $a \in S$, writes $\mathcal{L}^2_a$ or $\mathcal{L}^2_e(S)$.

It is easy to prove that the following:

Lemma 2.3 Let $S$ be a $\gamma$-right abundant semigroup with central idempotent $E(S)$. Then $\mathcal{L}^2_e$ is a $\mathcal{L}^2$-unipotent semigroup, where $e \in E(S)$. Dually, if $S$ is a $\gamma$-left abundant semigroup with central idempotent $E(S)$. Then $\mathcal{L}^2_a$ is a $\mathcal{L}^2$-unipotent semigroup.

Lemma 2.4 Let $S$ be a $\gamma$-right(left) abundant semigroup with central idempotent $E(S)$, and $\mathcal{L}^2(\mathcal{R}^2)$ is a right(left) congruence on $S$. Then $\mathcal{L}^2(\mathcal{R}^2)$ is a semilattice congruence on $S$.

Lemma 2.5 Let $\mathcal{L}^2$ is a right congruence on $\gamma$-right abundant semigroup $S$. Then $S$ is a $\gamma$-right abundant semigroup with central idempotents $E(S)$ if and only if $S$ is a strongly semilattice of $\mathcal{L}^2$-unipotent monoids.

Let $S$ be an $\gamma$-abundant semigroup. Denote the idempotents in $\mathcal{L}^2_a \cap E(S)$ and $\mathcal{R}^2_a \cap E(S)$ by $a^\dagger$ and $a^+, \text{ respectively}$. It is well known that any band $B$ can be expressed as a semilattice of rectangular bands $E_\alpha$, write $B = \bigcup_{\alpha \in \mathcal{Y}} E_\alpha$. If $e \in E_\alpha$, then we denote $E_\alpha$ by $B(e)$. Also, if $E_\alpha E_\beta \subseteq E_\alpha$, then we denote this inclusion by $E_\alpha \leq E_\beta$.

The following lemma give some basic properties of $\gamma$-quasi-adequate semigroups

Lemma 2.6 Let $S$ be a $\gamma$-quasi-adequate semigroup, and $a, b \in S$. If $a = be$ for some $e \in E(S)$, then $E(a^+^\dagger) \leq E(e)$.

Proof. Since $a = be$ for some $e \in E(S)$, we have $a = ae$ and $a^+ = a^\dagger e$ by $a\mathcal{L}^2a^\dagger$. It implies that $E(a^+^\dagger) \leq E(e)$.

Lemma 2.7 Let $S$ be a $\gamma$-adequate semigroup of type C with a normal band $E(S)$. Then

$$\delta = \{(a, b) \in S \times S | a = ebf \text{ for some } e \in E(b^+) \text{ and } f \in E(b^+)\}$$

is the minimum good congruence on $S$ such that $S/\delta$ is a $\gamma$-adequate semigroup of type C.

Definition 2.8 (i) A band $B$ is called a left(right) quasi-normal band if $B$ satisfies the identity $xyz = xzy [xyz = xzy]$. 
(ii) A band $B$ is called a [left,right] normal band if $B$ satisfies the identity $[xyz = xzy; xyz = yxz][xyzw = xzyw];

(iii) A band $B$ is called a [left] regular band if $B$ satisfies the identity $xy = xyx[xy = yxy].$

Lemma 2.9 The following statements are equivalent for a band $B$:

(i) $B$ is a left [right] quasi-normal band;
(ii) $B$ is the spined product of a left[regular] regular band and a right [left] normal band;
(iii) $R[L]$ is a left[regular] regular band on $B$ while $L[R]$ is a right [left] normal band congruence on $B.$

Lemma 2.10 Let $S$ be a $\#^\ast$-abundant semigroup of type C, and $E(S)$ a left quasi-normal band. Then the relation

$$\eta = \{(x, y) \in S \times S | x = yf \text{ for some } f \in E(y\ast)\}$$

is the minimum good congruence on $S$ such that $S/\eta$ is an $R^{2}$-unipotent semigroup.

3 $\lambda(\rho)$-$\#^\ast$-abundant semigroups

In this section, we introduce the concept of $\lambda(\rho)$-$\#^\ast$-abundant semigroups. The characterization theorem on perfect $\#^\ast$-abundant semigroups of type C is obtained.

Let $\lambda_e$ denote a left translation of a band $B$, that is, $\lambda_e(a) = ex$ for any $x \in B$, where $e$ is a given idempotent in $B$. Obviously, $\lambda_e$ maps $B$ into $B$. If the band $B$ is a left quasi-normal band, then $exy = exey$ for any $x, y$. Thus, we follow that $\lambda_e(xy) = exy = exey = \lambda_e(x)\lambda_e(y)$. This shows that $\lambda_e$ is an endomorphism of $B$ into itself. Dually, we can define a right translation $\rho_e$ of the band $B$, and $\rho_e$ is also an endomorphism of $B$ into itself.

We begin with the following definition.

Definition 3.1 (i) $S$ is said to be $\lambda$-$\#^\ast$-abundant if the left translation $\lambda_e$ on $S$, determined by $e$, is an endomorphism of $S$, where $e \in E(S)$.

(ii) $S$ is said to be $\rho$-$\#^\ast$-abundant if the right translation $\rho_e$ on $S$, determined by $e$, is an endomorphism of $S$, where $e \in E(S)$.

(iii) $S$ is said to be $(\lambda, \rho)$-$\#^\ast$-abundant if $S$ is both $\lambda$-$\#^\ast$-abundant and $\rho$-$\#^\ast$-abundant.

Lemma 3.2 (i) If $S$ is a $\lambda - \#^\ast$-abundant semigroup, then $S$ is the $\#^\ast$-abundant semigroup whose all idempotents $E(S)$ are a left quasi-normal band. Dually, if $S$ is a $\rho - \#^\ast$-abundant semigroup, then $S$ is the $\#^\ast$-abundant semigroup whose all idempotents $E(S)$ are a right quasi-normal band.

(ii) If $S$ is a $(\lambda, \rho) - \#^\ast$-abundant semigroup, then $S$ is the $\#^\ast$-abundant semigroup whose all idempotents $E(S)$ are a normal band.

Proof. We only need to prove (i) since (ii) is immediately deduced by (i). Let $S$ is a $\lambda - \#^\ast$-abundant semigroup. For all $e, f, g \in E(S)$, we have $efg = efeg$ since the left translation $\lambda_e$ is an endomorphism on $S$. Let $g = f$, then $ef = eff = efef$. This shows that $E(S)$ is a left quasi-normal band, and whence $S$ is the $\#^\ast$-abundant semigroup whose all idempotents $E(S)$ are a left quasi-normal band. Dually, we can show that if $S$ is a $\rho - \#^\ast$-abundant semigroup, then $S$ is the $\#^\ast$-abundant semigroup whose all idempotents $E(S)$ are a right quasi-normal band.

Lemma 3.3 Let $S$ is a $\lambda - \#^\ast$-abundant semigroup. Then $E(ea)^* \leq E(e)$ for any $a \in S$ and $e \in E(S)$.

Proof. Assume that $S$ is a $\lambda - \#^\ast$-abundant semigroup. Then for any $a \in S$ and $e \in E(S)$, $ea = eaea^*$. By Lemma 2.6 and the definition of $L^\ast$, we have $(ea)^* = (ea)^* ea^*$. This means $E(ea)^* \leq E(e)$.

Lemma 3.4 Let $S$ is a $\#^\ast$-abundant semigroup with a left quasi-normal band $E(S)$, and satisfy the identity $ea = eae$ for all $e \in E(S/\eta)$ and $a \in S/\eta$, where $\eta$ is a relation on $S$ defined in Lemma. Then $S$ is a $\lambda$-$\#^\ast$-abundant semigroup.

Proof. Suppose that $S$ is a $\#^\ast$-abundant semigroup with a left quasi-normal band $E(S)$, and satisfy the identity $ea = eae$ for all $e \in E(S/\eta)$ and $a \in S/\eta$, where $\eta$ is a relation on $S$ defined in
Lemma 2.10. Then \((eab)\eta = (eaeb)\eta\) for all \(a, b \in S\) and \(e \in E(S)\), and so \(eab = eaebf\) for some \(f \in E(eaeb)^*\). Because \(E(S)\) is a left quasi-normal band, we imply that

\[
\lambda_e(ab) = eab = eab^* = eaeb^* = eaeb(eaeb)^* f(eaeb)^* b^* = eaeb(eaeb)^* b^* = eaeb = \lambda_e(a)\lambda_e(b).
\]

This means that \(\lambda_e\) is a endomorphism of \(S\), and so \(S\) is a \(\lambda - \sharp\) abundant semigroup.

Equipped with the above results and its dual, we now give a characterization for \((\lambda, \rho) - \sharp\) abundant semigroups of type \(C\).

Theorem 3.5 The following statements are equivalent:

(i) \(S\) is a \(\lambda - \sharp\) abundant semigroups of type \(C\);

(ii) \(S\) is a \(\lambda - \sharp\) abundant semigroups whose all idempotents \(E(S)\) are a left quasi-normal band of type \(C\), and \(E(ea)^* \leq E(e)\) for any \(a \in S\) and \(e \in E(s)\);

(iii) \(S\) is a \(\lambda - \sharp\) abundant semigroups whose all idempotents \(E(S)\) are a left quasi-normal band of type \(C\), and satisfy the identity \(ea = eae\) for all \(e \in E(S/\eta)\) and \(a \in S/\eta\), where \(\eta\) is a relation on \(S\) defined in Lemma 2.10.

As an application of above results, we establish the structure theorem of perfect \(\sharp\) abundant semigroups of type \(C\).

Theorem 3.6 The following statements are equivalent:

(i) \(S\) is a \((\lambda, \rho) - \sharp\)-abundant semigroup of type \(C\);

(ii) \(S\) is a both perfect \(\sharp\)-rpp and perfect \(\sharp\)-lpp semigroup of type \(C\);

(iii) \(S\) is of type \(C\) and a spined product of a \(C-\sharp\)-semigroup and a normal band and \(S\);

(4) \(S\) is of type \(C\) and a strong semilattice of rectangular unipotent semigroup, that is, \(S\) is a perfect \(\sharp\)-abundant semigroup of type \(C\).

Proof. (1)\(\Rightarrow\) (2). Assume that (1) holds. Then \(E(S)\) is a normal band, and \(S/\delta\) is a \(C-\sharp\)-abundant.

References


