The Extended Legendre-Stirling Numbers of the First Kind

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**Abstract.** The Legendre-Stirling numbers of the first kind $P^s_n^{(j)}$ are defined by the coefficients of Taylor expansion of the function \( \frac{x(x-2)(x-6)\cdots(x-(n-1)n)}{1} \) by Andrews and Littlejohn (see "A combinatorial interpretation of the Legendre-Stirling numbers", Proc. Amer. Math. Soc, 137: 2581-2590, 2009). In this paper, two new kinds of numbers $P^{(-j)}_{n-j}$ and $P^{(j)}_{n+j}$ are proposed with the coefficients of Laurent expansion of the function \( \left(x(x-2)(x-6)\cdots(x-(n-1)n)\right)^{-1} \), which are called the extended Legendre-Stirling numbers of the first kind. Several properties of the two new sequences are proved, such as the recurrence relations, vertical recurrence relation, forward difference. Also, this paper shows a relational expression of the Legendre-Stirling numbers of the extended first and second kinds.

1 Introduction

The Legendre-Stirling numbers of the second kind $PS_n^{(j)}$ were first introduced in 2002 as a result of a problem involving the spectral theory of powers of the classical second-order Legendre differential expression by Everitt, Littlejohn and Wellman (see [1]). Specifically, these numbers are the coefficients of integral composite powers of the Legendre expression in Lagrangian symmetric form. Quite remarkably, they share many similar properties with the classical Stirling numbers of the second kind which are the coefficients of integral powers of the Laguerre differential expression.

The Legendre-Stirling numbers of the first kind $P^s_n^{(j)}$ were defined by Andrews and Littlejohn[2] in 2009. The properties of the Legendre-Stirling numbers of the first kind, such as the recurrence relations, were studied by Egge in 2010[3].

From Ref.[2], we know that $n,j$ are non-negative integer in the Legendre-Stirling numbers. In this paper, with reference to the “sum function” of the Legendre-Stirling numbers of the first kind, here the “sum function” is \( \langle x \rangle_n = x(x-2)(x-6)\cdots(x-(n-1)n) \). In this paper, we introduce a new function \( \langle x \rangle_{-n} = [x(x-2)(x-6)\cdots(x-(n-1)n)]^{-1} \). A new type of sequence is defined by the coefficient of Laurent expansion of the new function. These numbers have similar recurrence relations to the Legendre-Stirling numbers of the first kind which are called the extended Legendre-Stirling numbers of the first kind. In fact, it can be seen as an extension of the domain of the Legendre-Stirling numbers of the first kind, that is, the range of $n,j$ are expanded to integers.

2 The extension of the Legendre-Stirling numbers of the first kind

Andrews and Littlejohn[4] defined the Legendre-Stirling numbers of the first kind $PS_n^{(j)} (0 \leq j \leq n)$ via \( \langle x \rangle_n = \sum_{j=0}^{n} P^s_n^{(j)} x^j \), where \( \langle x \rangle_0 = 1 \). We can see that the numbers
$P_{n,j}^{(j)}$ ($0 \leq j \leq n$) are the coefficients of Taylor series expansion of the function $\langle x \rangle_n$. The sequence $\{ \langle x \rangle_n \}_{n=0}^{\infty}$ satisfy the relationship $\langle x \rangle_n = (x - (n-1)n)\langle x \rangle_{n-1}^2$ $(n \geq 1)$. And the Legendre-Stirling numbers $\{ P_{n,j}^{(j)} \}$ satisfy the following triangular recurrence relation [2]: $P_{n,j}^{(j)} = P_{n-1,j}^{(j-1)} - n(n-1)P_{n-1,j+1}^{(j)}$, $P_{n,0}^{(j)} = 0(n > 0)$, $P_{0,j}^{(j)} = 0$ $(0 < j \leq n)$, $P_{0,0}^{(0)} = 1$. In order to extend the Legendre-Stirling numbers of the first kind, we introduce the function

$$\langle x \rangle_{-n} = \left( x(x-2)(x-6)\cdots(x-(n-1)n) \right)^{-1},\ n \geq 1; \langle x \rangle_0 = 1. \quad (1)$$

It is easy to see that the sequence $\{ \langle x \rangle_{-n} \}_{n=0}^{\infty}$ satisfy the relationship $\langle x \rangle_{-n} = (x - (n-1)n)^{-1} \langle x \rangle_{-(n-1)}$ $(n \geq 1)$. Now we show the Laurent series expansion of $\langle x \rangle_{-n}$ $(n \geq 1)$.

(1) Let $X_1 = \{ x \mid 0 < x < 2 \}$. We have

$$\langle x \rangle_{-n} = \frac{1}{x} (-1)^{n-1} \prod_{i=1}^{n} \left[ \frac{1}{i(i+1)} \left( 1 - \frac{x}{i(i+1)} \right) \right]^{-1} \frac{1}{x} (-1)^{n-1} \prod_{i=1}^{n-1} \left[ \frac{1}{i(i+1)} \left( \sum_{k=0}^{\infty} \frac{x^k}{(i+1)^k} \right) \right] = (-1)^{n-1} \prod_{i=1}^{n-1} \left[ \frac{1}{i(i+1)} \left( \sum_{j=1}^{\infty} \frac{x^j}{(i+1)^j} \right) \right] \quad (2)$$

(2) Let $X_2 = \{ x \mid (n-1)n < x < +\infty \}$. We have

$$\langle x \rangle_{-n} = x^{-n} \prod_{i=1}^{n-1} \left[ 1 - \frac{l(l+1)^{-1}}{x} \right]^{-1} = x^{-n} \prod_{i=1}^{n-1} \left[ \sum_{k=0}^{\infty} \frac{l(l+1)^k}{x} \right]^k = x^{-n} \prod_{i=1}^{n-1} \left[ \sum_{k=0}^{\infty} \frac{x^k}{(i+1)^k} \right] \sum_{j=0}^{\infty} \frac{(1 \cdot 2)^j \cdot (2 \cdot 3)^j \cdots ((n-1)n)^j \cdot x^{-j}}{j!}.$$  

In this paper, we define the extended Legendre-Stirling numbers of the first kind via the coefficients of the Laurent series expansion of $\langle x \rangle_{-n}$ $(n \geq 0)$.

**Definition 1.** For all $n \geq 0$, $X_1 = \{ x \mid 0 < x < 2 \}$. In the range of $X_1$, the Laurent series expansion of the function $\langle x \rangle_{-n}$ is given by

$$\langle x \rangle_{-n} = \sum_{j=-1}^{\infty} P_{n,j}^{(j)} x^j, \quad (3)$$

where the coefficients $P_{n,j}^{(j)}$ $(j \geq -1)$ are called the extended Legendre-Stirling numbers of the first kind. We can see that

(1) for $n=0$, since $\langle x \rangle_{-n} = 1$, we get $P_{0,0}^{(0)} = 1, P_{0,j}^{(j)} = 0 (j \neq 0)$;  

(2) for $n=1$, since $\langle x \rangle_{-1} = \frac{1}{x}$, we get $P_{0,-1}^{(-1)} = 1, P_{0,j}^{(j)} = 0 (j \neq -1)$;  

(3) for $n>1, j=-1$, from the equation (2), we get $P_{n,-1}^{(-1)} = (-1)^{n-1} \prod_{i=1}^{n-1} \frac{1}{i(i+1)}$.  

(4) for $n \geq 0, j < -1$, the numbers $\{ P_{n,j}^{(j)} \}$ can not be found in $X_1$, so we get $P_{n,j}^{(j)} = 0 (n \geq 0, j < -1)$.

**Definition 2.** For all $n \geq 0$, $X_2 = \{ x \mid (n-1)n < x < +\infty \}$. In the range of $X_2$, the Laurent series expansion
of the function \( \langle x \rangle_n \) is given by
\[
\langle x \rangle_n = \sum_{j=0}^{\infty} P_{n-1}^{(j)} x^j,
\]
where the coefficients \( P_{n-1}^{(j)} \) \((j \geq n \geq 0)\) are called the extended Legendre-Stirling numbers of the first kind. We can see that

1. for \( n=0 \), since \( \langle x \rangle_0 = 1 \), we get
\[
P_{0}^{(0)} = 1, P_{0}^{(j)} = 0 \; (j > 0);
\]
2. for \( n = 1 \), since \( \langle x \rangle_1 = \frac{1}{x} \), we get
\[
P_{1}^{(-1)} = 1, P_{0}^{(-j)} = 0 \; (j > 1);
\]
3. for \( 0 < j < n \), the numbers \( \{ P_{n-1}^{(j)} \} \) can not be found in \( X_2 \), so we can get \( P_{n-1}^{(j)} = 0 \; (0 < j < n) \).

3 The properties of the extended Legendre-Stirling numbers of the first kind

When \( n=0, n=1 \), we have got equations (4), (5), (6), (7) about the extended Legendre-Stirling numbers of the first kind. For \( n>1 \), the following results are clear proved about the numbers.

**Theorem 1.** If \( n > 1 \), then \( P_{n-1}^{(j)} = (-1)^{n-1} \prod_{i=1}^{n-1} \frac{1}{i(i+1)} \left[ \sum \prod_{i=1}^{n-1} \left( \frac{1}{i(i+1)} \right)^h \right] \), where \( * \) denoted \( h_i \geq 0 \) and \( \sum_{i=1}^{n-2} h_i = j + 1(n > 1, j \geq 0) \).

**Proof.** From definition 1, we have
\[
\sum_{j=0}^{\infty} P_{n-1}^{(j)} x^j = \langle x \rangle_n = \frac{1}{x} \left( -1 \right)^{n-1} \prod_{i=1}^{n-1} \frac{1}{i(i+1)} \left[ \sum \prod_{i=1}^{n-1} \left( \frac{1}{i(i+1)} \right)^h \right] x^j.
\]

**Theorem 2.** For \( n > 1 \), the extended Legendre–Stirling numbers of the first kind \( \{ P_{n-1}^{(j)} \}_{j=1}^{\infty} \) satisfy the following triangular recurrence relation
\[
P_{n-1}^{(j)} = P_{n-1}^{(j-1)} - n(n+1) P_{n-1}^{(j)}.
\]

**Proof.** (1) For \( j = -1 \), since \( P_{n-1}^{(-1)} = (-1)^{n-1} \prod_{i=1}^{n-1} \frac{1}{i(i+1)}, P_{n-1}^{(-2)} = 0 \), it can be got
\[
P_{n-1}^{(-2)} = -n(n+1) P_{n-1}^{(-3)}.
\]

Thus we have the equation (10).

2. for \( j \geq 0 \), from the equations (1) and (3), we have
\[
\sum_{j=0}^{\infty} P_{n-1}^{(j)} x^j = \sum_{j=1}^{\infty} P_{n-1}^{(j)} x^{j+1} - n(n+1) \sum_{j=1}^{\infty} P_{n-1}^{(j)} x^j,
\]
By equation (10), we have
\[ P_{n-1}^{(-)} x^{-1} + \sum_{j=0}^{\infty} P_{n-1}^{(-)} x^j = \sum_{j=0}^{\infty} P_{n-1}^{(-)} x^j - n(n+1) P_{n-1}^{(-)} x^{-1} - n(n+1) \sum_{j=0}^{\infty} P_{n-1}^{(-)} x^j. \]

So we can get
\[ P_{n-1}^{(-)} = n(n+1) P_{n-1}^{(-)} x^{-1} \quad (j \geq 0). \]

This completes the proof of the theorem.

Theorem 3. For \( n, j \in \mathbb{Z} \) \((1 \leq n \leq j)\), the extended Legendre–Stirling numbers of the first kind \( \{ P_{n-1}^{(-)} \}_{j=n}^{+\infty} \) satisfy the following triangular recurrence relation:
\[ P_{n-1}^{(-)} = P_{n-1}^{(-)} - n(n+1) P_{n-1}^{(-)}. \]

Proof. From equations (1) and (6), we have
\[ \sum_{j=0}^{\infty} P_{n-1}^{(-)} x^j = \sum_{j=0}^{\infty} P_{n-1}^{(-)} x^j - n(n+1) \sum_{j=0}^{\infty} P_{n-1}^{(-)} x^j. \]

Since \( P_{n-1}^{(-)} = 0 \), so it can be got
\[ \sum_{j=0}^{\infty} P_{n-1}^{(-)} x^j = \sum_{j=0}^{\infty} P_{n-1}^{(-)} x^j - n(n+1) \sum_{j=0}^{\infty} P_{n-1}^{(-)} x^j. \]

Hence, we have
\[ P_{n-1}^{(-)} = P_{n-1}^{(-)} - n(n+1) P_{n-1}^{(-)}. \]

Lemma 1. For all \( n, j \in \mathbb{N} \), \( \{ P_{n}^{(-)} \} \) satisfy the following relation
\[ P_{n}^{(-)} = \sum_{i_{1}+i_{2}+\ldots+i_{n}=n-j} (1 \times 2^{i_{1}} \cdot 2 \times 3^{i_{2}} \cdot \ldots \cdot (j+1)^{i_{n}}). \]  

(11)

Theorem 3. For all \( n, j \in \mathbb{N} \) \((0 < n \leq j)\), we have
\[ P_{n}^{(-)} = \sum_{i_{1}+i_{2}+\ldots+i_{n}=n-j} (1 \times 2^{i_{1}} \cdot 2 \times 3^{i_{2}} \cdot \ldots \cdot (n-1)n^{i_{n}}). \]  

(12)

Proof. For all \( n \in \mathbb{N} \), the sequence \( \{ P_{n}^{(-)} \}_{j=n}^{+\infty} \) satisfy the following horizontal generating function
\[ F_{n}(x) = \sum_{j=0}^{+\infty} P_{n}^{(-)} x^j. \]

So, we can see that
\[ \sum_{j=0}^{+\infty} P_{n}^{(-)} x^j = F_{n}(x) = \frac{1}{x(x-2)(x-2 \times 3) \cdots (x-(n-1)n)} = x^{-n} \prod_{l=1}^{n-1} \left( 1 - \frac{l(l+1)}{x} \right)^{-1} = x^{-n} \prod_{l=0}^{n-1} \sum_{i_{1}+i_{2}+\ldots+i_{n}=j-n} (1 \times 2^{i_{1}} \cdot 2 \times 3^{i_{2}} \cdot \ldots \cdot (n-1)n^{i_{n}} x^{-i}). \]

So,
\[ P_{n}^{(-)} = \sum_{i_{1}+i_{2}+\ldots+i_{n}=j-n} (1 \times 2^{i_{1}} \cdot 2 \times 3^{i_{2}} \cdot \ldots \cdot (n-1)n^{i_{n}}). \]

Comparing (11) and (12), we find that

Corollary 1. \( P_{n}^{(-)} = P_{j-1}^{(-)}. \)

Lemma 2. \( \{ P_{n}^{(-)} \} \) satisfy the vertical recurrence relation:
\[ P_{n}^{(-)} = \sum_{k=j}^{n} P_{k}^{(-)} (j+1)^{n-k} (n, j \in \mathbb{N}). \]

In particular, \( P_{0}^{(-)} = 0, P_{0}^{(-)} = 0, P_{0}^{(-)} = 1. \)
By Corollary 1 and Lemma 2, we can get the following conclusions:

**Theorem 4.** For all \( n, j \) (\( 0 < n \leq j \)), \( \{P_{s_{(n-j)}}^{(j)}\} \) satisfy the vertical recurrence relation:

\[
P_{s_{(n-j)}}^{(j)} = \sum_{k=0}^{j} P_{s_{(n-k)}}^{(k+n)} [(n-1)^{-1}]^{j-k}.
\]

For the Legendre–Stirling numbers of the second kind, we note that \( P_{s_{(1)}} = 2^{n-1} \). From Corollary 1, we can get the following series of corollaries.

**Corollary 2.** \( P_{s_{(2)}} = P_{s_{(1)}} = 2^{n-1} (n \geq 1) \).

From the unimodality of the Legendre–Stirling numbers of the second kind (see Ref.[6]), we can obtain the following conclusion.

**Corollary 3.** For \( j \in \mathbb{N}^* \), the sequence \( \{P_{s_{(n-j)}}^{(j)}\} \) satisfy the unimodality.

Since \( \{P_{s_{(n-j)}}^{(j)}\} \) satisfy \( P_{s_{(n-i)}} \equiv 0 \left( \text{mod } 2^n \right) \) (\( 0 \leq i \leq n \)), so we have that

**Corollary 4.** \( P_{s_{(n-j)}}^{(j)} \equiv 0 \left( \text{mod } 2^n \right) \) (\( 0 \leq n \leq j \)).

**Lemma 3.** For \( k \in \mathbb{N}^* \), we have \( \Delta^j \left( \frac{P_{s_{(n-j)}}^{(j)}}{2^n} \right) \geq 0 \) (\( n \geq j \)).

By Corollary 1 and Lemma 3, we can get the following conclusion.

**Corollary 5.** For \( k \in \mathbb{N}^* \), we have \( \Delta^j \left( \frac{P_{s_{(n-j)}}^{(j)}}{2^n} \right) \geq 0 \) (\( n \geq j \)).

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**References**


