Application of Structured Low Rank Approximation in Modal Analysis

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Abstract. The quality of measured vibration response signals is adversely affected by noise in modal testing. This situation often leads to serious difficulties in estimating the modal parameters with proper accuracy. This paper presents a signal de-noising method for measured impulsive response functions (IRFs) based on structured low rank approximation (SLRA) so as to improve the accuracy of the modal parameters identification. The proposed method is verified by simulation study of a cantilever beam. The results show that this method can eliminate noise from measured IRFs efficiently, and significant improvement in modal parameters estimation could be obtained based on the filtered IRFs.

Introduction

Modal parameter estimation is the process of determining modal parameters of structures from experimental data. Accurate estimate of the modal parameters, in addition to being utilized in a typical modal-based damage detection method, can also provide a basis for finite element updating and validation.

Conventional modal identification techniques employed in laboratories are often based on estimating a set of frequency response functions (FRFs), or on the corresponding impulse response functions (IRFs), relating the applied force (input) and corresponding response (output) at several pairs of points along the structure with enough high spatial and frequency resolution. The modal parameters of civil structures, however, are often required to be estimated exclusively from output signals because dynamic testing of large structures, such as bridge, offshore platforms, can only be performed in the field and the application of excitation force is usually not practicable as it is technically and/or economically prohibitive. Extracting modal parameters from output data has been a subject of constant improvement and enhancements, and the reader is referred to the subject textbooks for a detailed treatise on various modal identification methods [1].

As the basis of many output-only time domain methods, the complex exponential (CE) algorithm, has historically been derived from Prony's method [1]. The method simply takes an impulse response function (IRF), which is composed of summation of damped exponentials, and estimates the amplitude, frequency and damping of the individual terms in the summation. The CE method has become one of the most popular methods in modern commercial software for experimental modal analysis. However, in comparison with other modern modal identification methods, such as the eigensystem realization algorithm (ERA) [2] and stochastic subspace identification (SSI) method [3], a drawback of the traditional CE method is that it does not have a noise rejection mechanism in the identification process and is sensitive to the noise embedded in the signal.

The traditional methods to improve the accuracy of parameters identification in noisy situations are based on over-determined system, and the distinguishing procedure (such as stability diagrams) could be used to get the true system modes [4-6]. However, it is still difficult to sort out noise modes, as the distinguishing procedure relies on the user's experience and judgment. Therefore, it is of great
importance to develop a method that can overcome this problem to estimate the modal parameters
with high accuracy.

This article proposed a data processing procedure to remove much noise from the measured IRF
before applying a time domain technique for modal parameters identification. In order to remove
noise from the measured IRF to yield a filtered IRF, a structured low rank approximation (SLRA)
method for the Hankel matrix is carried out. Once the filtered IRF is obtained, the CE method can be
applied to extract the modal parameters [1]. A simulation study of cantilever beam will be used to
demonstrate the performance of the proposed scheme and illustrate the procedure as well.

Theory

IRF and CE Method

For facilitating our later presentation, a brief description of the IRF of an N-degree-of-freedom
(N-DOF) system and the essential of CE method are given below.

A general expression for any FRF of an N-DOF dynamic system can be written as

$$ F(\omega) = \sum_{k=1}^{N} \left( \frac{A_k}{a_k \xi_k + i(\omega - a_k \sqrt{1 - \xi_k^2})} + \frac{A_k^*}{a_k \xi_k + i(\omega + a_k \sqrt{1 - \xi_k^2})} \right) $$

where \(a_k\) and \(\xi_k\) are the natural frequency and damping ratio, respectively, of the \(k\)th mode, \(A_k\) a
complex amplitude, and the superscript '*' denotes the complex conjugate operator. By taking the
inverse Fourier transform of Eq. (1), the corresponding IRF is obtained as

$$ h(t) = \sum_{k=1}^{2N} A_k e^{s_k t} $$

where \(s_k = -\omega_k \xi_k + i\omega_k\). If the original FRF has been obtained in a discrete form, the resulting discrete
IRF can be expressed as

$$ h_l = h(l\Delta t), \quad l = 0, 1, 2, \ldots $$

where \(\Delta t\) is the time interval. From Eqs. (2) and (3), we have

$$ h_l = \sum_{k=1}^{2N} A_k V_k^l $$

where \(V_k = e^{s_k l}\). The above equation is a non-linear function of the unknown modal parameters.
However, once \(V_k\) (i.e. modal frequencies and damping ratios) are known, Eq. (4) becomes linear
with respect to the remaining unknown modal parameters \(A_k\).

Because an N-DOF dynamic system is mathematically equivalent to a linear differential equation of
order 2N, the IRF sequence \{ \(h_l\) \} must satisfy a linear difference equation:

$$ h_{2N+n} + \sum_{m=0}^{2N-1} \beta_n h_{n+m} = 0, \quad n = 0, 1, \ldots $$

where \(\beta_n\), \(m = 0, \ldots, 2N-1\) are real constant coefficients. Inserting Eq. (5) into Eq. (4) leads to:

$$ \sum_{k=1}^{2N} \beta_n V_k^n = 0 $$

where \(\beta_{n}\) has been set equal to 1, and the roots of the polynomial are \(V_1, V_2, \ldots, V_{2N}\). Thus, the values of
the \(\beta_n\) determine the values of \(V_k\), and hence the system natural frequencies and damping ratios. In
solving \(\beta_n\) from the IRF data, the original CE method has been based on the successive applications
of Eq. (5) for 2N times. Taking \(n = 0, \ldots, 2N-1\) with Eq. (5) results in a full set of 2N equations.
\[
\begin{bmatrix}
    h_0 & h_1 & h_2 & \cdots & h_{2N-1} \\
    h_1 & h_2 & h_3 & \cdots & h_N \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    h_{2N-1} & h_{2N} & h_{2N+1} & \cdots & h_{3N-2}
\end{bmatrix}
\begin{bmatrix}
    \beta_0 \\
    \beta_1 \\
    \vdots \\
    \beta_{2N-1}
\end{bmatrix} =
\begin{bmatrix}
    h_{2N} \\
    \vdots \\
    h_{3N-1}
\end{bmatrix}
\]  
\tag{7}

or

\[
H\beta = \tilde{h}
\]

where \( H \in \mathbb{R}^{2N \times 2N} \) is a square matrix with Hankel structure, which is referred to as a matrix with constant (positive sloping) skew-diagonals. From Eq. (8), we can obtain the unknown polynomial coefficients

\[
\beta = -H^t\tilde{h}
\]

\tag{9}

Structured Low Rank Approximation

An IRF sequence of an N-degree of freedom system \( \{h_\ell\}, \ell = 0, \ldots, s \), can be sequentially filled to form a rectangular Hankel matrix \( H_{pq} \in \mathbb{R}^{p \times q} \), with \( p, q \geq 2N \) as

\[
H_{pq} =
\begin{bmatrix}
    h_0 & h_1 & \cdots & h_{s-1} \\
    h_1 & h_2 & \cdots & h_s \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{p-1} & h_p & \cdots & h_s
\end{bmatrix}
\]  
\tag{10}

where \( s = p + q - 2 \).

In mathematical terms, the signal-noise separation problem here is a SLRA problem [7-9], which can be summarized by the following steps:

1. Use the SVD technique with an appropriate value of rank \( r \) estimated from the data to obtain a low rank approximation matrix \( \hat{H} \).

2. Rebuild a Hankel matrix \( \hat{H} \) from \( \hat{H} \) by replacing all elements of each anti-sub diagonal by the arithmetic average of the elements along the anti-sub diagonal.

3. Steps 1 and 2 are alternated iteratively until a convergence test has been met.

Simulation Study

The performance of the proposed method is investigated by a cantilever beam modeled by ANSYS commercial software. The length of the steel beam is 1.5m, and the section is 57×25mm². The first two natural frequencies are 8.35 and 52.32Hz, respectively. A proportional damping matrix is taken, and the damping ratio \( \xi_1 \) and \( \xi_2 \) equal to 0.002 and 0.003. Exciting the beam by an impulsive load on the free end, the corresponding acceleration signals with sampling rate 200 Hz are recorded. Without any loss of generality, only a segment about 5 seconds with 1024 sample points of the free end would be taken for the later analysis. Corrupted IRF is referred to as “measured” IRF, which is generated by adding a Gaussian white noise to the noise-free IRF. Shown in Fig. 1 is the exact related FRF and that with 5% white noise signal.
In SLRA method, the appropriate value of rank of the Hankel matrix should be known in advance. The plot of normalized singular values is used to choose the appropriate rank when the normalized singular values approach an asymptote. In this section, the 1024-point signal is referred to as the measured IRF, from which the corresponding 513x512 largest Hankel matrix could be produced. The normalized singular values of the Hankel matrix is shown in Fig. 2. Because the fifth normalized singular value drops to a much smaller value, the noise threshold is set to 4.

Working on the Hankel matrix using SLRA based on rank reduction 4, we obtain an excellent result of the noise removal (see Fig. 3). The filtered FRF has 2 sharp peaks within the frequency range 0-100 Hz, and the visible level of noise at the measured FRF has been successfully removed.

After carrying out the noise cancellation to obtain the filtered IRFs, CE method is applied for estimating modal parameters. The first two modal frequencies and damping ratios estimated from the exact, measured and filtered signals are listed in Table 1, where the values of finite element model (FEM) are also included for comparison. The two modal frequencies and damping ratios obtained from the exact data basically agree with those of the FEM, recognizing that discrepancy always exists between the true modal parameter and it’s estimated from truncated signals.
As exhibited in Table 1, the two order modal parameters estimated from the filtered IRFs are in excellent agreement with that from the exact IRFs, respectively. However, the modal frequencies and damping ratios estimated from the measured IRF are worse, and the second modal parameters are not identified due to the noise.

Table 1. The first two modal frequencies and damping ratios estimated from different signals.

<table>
<thead>
<tr>
<th>Mode</th>
<th>FEM</th>
<th>Exact</th>
<th>Measured</th>
<th>Filtered</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f_1$ (Hz)</td>
<td>$\xi_1$ (%)</td>
<td>$f_2$ (Hz)</td>
<td>$\xi_2$ (%)</td>
</tr>
<tr>
<td>1</td>
<td>8.35</td>
<td>0.20</td>
<td>8.34</td>
<td>0.22</td>
</tr>
<tr>
<td>2</td>
<td>52.32</td>
<td>0.30</td>
<td>52.45</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Concluding Remarks

Due to the noise in measured vibration response signals, accurately identifying modal parameters has been a challenging task. This paper presents a data processing procedure to remove noise from measured IRFs based on structured low rank approximation so as to improve the accuracy of the modal estimation.

Simulation study of a cantilever beam was investigated. The modal parameters (frequencies and damping ratios) could be accurately identified using the proposed method. Significant improvement in modal parameters estimation could be obtained when the filtered, rather than measured, IRF was used.

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References


