Reasoning with Cardinal Directions in 3D Space Based on Block Algebra

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ABSTRACT

Spatial relations have been studied in various interrelated areas such as qualitative spatial reasoning, geographic information science, image retrieval, and others. Direction as one of the most important spatial relations has received much attention. In this paper, we introduce a new formalism, we name BCD calculus, for qualitative spatial reasoning with block cardinal direction relations between blocks in 3D space. Based on the theory of block algebra, the correlations between block cardinal direction relations in 3D and block algebra are explained, and the tractable subset of convex block cardinal direction relations is identified. We show that the consistency problem for convex block cardinal direction network can be solved in polynomial time, by translating qualitative networks of BCD calculus to qualitative network of block algebra and applying a suitable adaption of path consistency algorithm.

INTRODUCTION

As one of the most important spatial relations, the direction relation describe show.

Spatial object is placed relative to one another in a certain reference frame. In recent years, the study on direction has received much attention in the areas of GIS, AI, robotics, spatial database and etc. The research working on direction relations
mainly focuses on the aspects of representation and reasoning. The results of studies on direction relations in 2D space are relatively perfect. The cardinal direction in 3D space has become a hot and difficult topic with the development of science and technology and newly-emergent requirements of real application. Most researchers in qualitative spatial reasoning have made their achievements in representing and reasoning of direction relations in 3D. The 2D-orientation double cross model has been extended into 3D in [1]. The planar cardinal direction was extended into 3D space in [2] and a new model called TCD(three-dimensional cardinal direction) was obtained, in which two novel ways were proposed to compute the inverse and composition of basic TCD relations and an O(n4) algorithm was given for checking the consistency of basic TCD relations over simple blocks. Li et al. described qualitative 3D space by defining 3D spatial topological and directional relationships between objects, and explored spatial reasoning in 3D space by giving the composition tables of 3D spatial relationships, and constructed a weighted constraint graph of all objects to infer the locations of objects[3]. Chen et al. presented a model called the Objects Interaction Cube Matrix for determining cardinal directions between two volume objects in 3D space [4], which took into account the shapes of the volume objects and ensured the property of converseness of cardinal directions. The relationship between block algebra and 3D rectangular cardinal direction relations was draw out in [5], which was used for computing the inverse of the 3D rectangular cardinal directions. An approach to determining directional relations between gridded parts of the complex region in 2D was presented in [6], which was extended into 3D seamlessly.

The methods mentioned above mainly focused on reasoning with single or basic directions in 3D space. In order to improve the reasoning with cardinal direction relations, in this paper, the correlations between block cardinal direction relations in 3D and block algebra are explained, which are used to reason with spatial cardinal direction. Since the problem of directional spatial reasoning is NP-complete[7], on the basis of [8], we extend the convex tractable sub algebra into 3D space and identify the subset of convex block cardinal directions.

In this paper, we introduce BCD calculus and explain the relationship between block cardinal direction relations in 3D and block algebra. Based on the tractability of convex relations, the subset of convex block cardinal directions of BCD calculus is identified. In the end of the paper, we conclude that the consistency problem for convex block cardinal direction network can be solved in polynomial time.

RELATED WORKS

In this Section, we introduce BCD calculus, interval algebra and block algebra.
BCD Calculus

Consider the three-dimensional Euclidean space. Blocks are defined as non-empty and bounded sets of points in it. A block is basic if it is homeomorphic to the closed unite orb \(\{(x, y, z) | x^3+y^3+z^3 \leq 1\}\). A basic block is referred to as a cube in 3D space in the paper, whose twelve sides are parallel to three orthogonal axes, i.e. x, y and z axis, in 3D Euclidean space. In the sequel, blocks to which we refer are the cubes. All basic blocks form a set, denoted by \(\Gamma\).

Like projection model based on minimum bounding rectangles in 2D space, the reference block divides the 3D space into three layers in TCD[2], where there are 27 atomic cardinal directions, as shown in Figure 1 and Definition 1.

Definition 1. (Cardinal direction model in 3D) Let \(DR = \{Eu, NEu, SEu, Wu, SWu, NWu, Su, Nu, Cu, Em, NEm, SEMu, Wm, SWm, Nm, Sm, Nm, Em, NEm, SEm, Wd, SWd, NWd, Sd, Nd, Cd\}\) be a set including 27 atomic cardinal directions divided by the reference block \(O\). \(2DR\) is the power set of \(DR\). The direction relation of the primary block \(P\) with respect to the reference block \(O\) can be defined as \(\text{dir}(P, O) = C, C \subseteq 2DR\).

We call a tile or tiles in \(2DR\) direction block(s). For example, \(Su\) is a direction block and \(Su:Sm\) are direction blocks. In the sequel, we use \(P\) as primary block, and \(O\) as reference block, where \(P, O \in \Gamma\). The symbol \(\tau(O)\) refers to the subset of \(R^3\) corresponding to the tile(s) \(\tau\) w.r.t. the reference block \(O\).

Definition 2. (Block Cardinal Direction) A basic block cardinal direction (basic BCD-relation) is a set of pairs of blocks denoted by a direction blocks string \(\tau_1: \tau_2: \ldots : \tau_k, \tau_i \in DR, 1 \leq i \leq k\), with the following meaning: \(P \tau_1: \tau_2: \ldots : \tau_kO \Subset P^o \cap \tau_i(O) \neq \emptyset, \forall \tau_i \in \{\tau_1: \tau_2: \ldots : \tau_k\}\) and \(P^o \cap \tau(O) = \emptyset, \forall \tau \in DR \setminus \{\tau_1: \tau_2: \ldots : \tau_k\}\), where \(P^o\) is the interior of \(P\). A block cardinal direction (BCD-relation) is represented by a set \(C = \{c_1, \ldots , c_m\}\), where \(c_i\) is a direction blocks denoting a basic relation. If \(C\) is not a singleton then the relation is said to be disjunctive.

In this paper we study a restricted variant of cardinal direction, that which we name basic block cardinal direction calculus (BCD calculus), denoted by Bbcd, containing only 216 basic relations in 3D space (while in 2D, there are 36 basic rectangular cardinal direction [8]). The power set \(2^{\text{Bbcd}}\) is the full set of allowed relations of the BCD calculus.

The underlying set of cardinal direction is not closed under converse and composition in 2D, and is the same as in 3D. So, we define BCD calculus by using weak converse and weak composition as follows:

Definition 3. (BCD calculus) Let \((2^{\text{Bbcd}}, \cap, \circw, \text{won})\) be a BCD calculus on the set \(2^{\text{Bbcd}}\). Three operations, namely, intersection \((\cap)\), weak composition \((\circw)\) and weak converse \((\text{won})\), are the mapping: \(2^{\text{Bbcd}} \rightarrow 2^{\text{Bbcd}}\). The meaning of intersection is the same as the one in set theory. Let \(c, c_1, c_2\) and \(C_1, C_2\) be basicand disjunctive BCD relations respectively, the weak composition and converse
are defined as:
\[ C_1 \circ_w C_2 := \{ t \in B_{bcd} | t \cap (C_1 \circ_w C_2) \neq \emptyset \} \]

- (case of basic relations) \( C_1 \circ_w C_2 := \bigcup_{c_1 \in C_1, c_2 \in C_2} \{ t \in B_{bcd} | t \cap (C_1 \circ_w C_2) \neq \emptyset \} \);

- (case of disjunctive relations) \( w_{con}(C_1) := \{ c_2 \in B_{bcd} | \exists P, O, \text{dir}(P, O) = c_1, \text{dir}(O, P) = c_2 \} \);

- (case of disjunctive relations) \( w_{con}(D) := \bigcup_{c \in D} w_{con}(c) \).

### Interval Algebra and Block Algebra

Allen introduced the so-called Interval Algebra (IA for short) to model the relative position between events or temporal intervals [9]. Let \( B_1 \) be the set of 13 atomic relations of IA, i.e. \( B_1 = \{ b, m, o, f, s, e, d, f, d, o, i, s, m, b \} \). Balbiani et al. introduced the Rectangle Algebra (RA for short) by extending IA into 2D space [10]. Based on IA and RA, \( n \)-dimensional Block Algebra (BA for short) was defined as a set of relations, the block relations, together with the fundamental operations of composition, converse and intersection [11]. When \( n = 3 \), we get block algebra \( B_3 \) in 3D space. \( B_3 \) includes \( 13^3 \) atomic relations which constitute the exhaustive list of the permitted relations that can hold between two blocks. We call these relations basic BA-relations. We define \( B_3 \) as follows:

**Definition 4.** (Block Algebra) The Block Algebra \( B_3 = \{(r_x, r_y, r_z) | r_x, r_y, r_z \in B_1\} \), where \( r_x, r_y, r_z \) are the relations between intervals of primary block and reference block projected onto the \( x, y, z \) axis respectively in 3D space. \( P \) and \( O \) satisfy the atomic relation \( (r_x, r_y, r_z) \in B_3 \) if and only if, for every \( \pi \in \{x, y, z\} \), the interval \( P \downarrow \pi \) and \( O \downarrow \pi \) satisfy the atomic relation \( r \in B_1 \), where \( P \downarrow \pi \) and \( O \downarrow \pi \) represent the projection intervals onto the \( \pi \) axis.

For the atomic relation \( (r_x, r_y, r_z) \in B_3 \), the symbol \( \downarrow \pi \) represents the atomic relation \( r \in B_1 \). The full set of BA-relations is \( 2^{B_3} \). For any \( R \in 2^{B_3} \), \( \downarrow \pi \) represent the projection interval onto the \( \pi \) axis of \( R \), namely, \( R \downarrow \pi = \{ r \downarrow \pi | r \in R \} \).
Example 1. Let be the following BA-relation \( R = \{(m, fi, di), (m, o, di), (b, fi, di), (b, o, di)\} \), we have \( R \downarrow x = \{m, b\} \), \( R \downarrow y = \{fi, o\} \), \( R \downarrow z = \{di\} \).

Example 2. In Figure 2, two basic blocks \( P \) and \( O \) are represented, whose block relation satisfies atomic relation \((o, m, b)\), denoted by \( P(o, m, b)O \). The interval \( P \downarrow x \) overlaps the interval \( O \downarrow x \) on the x axis, the interval \( P \downarrow x \) and the interval \( O \downarrow x \) satisfy the atomic relation meets on the y axis, and \( P \downarrow x \) precedes \( O \downarrow x \) on the z axis.

For any \( R, T \in 2^{B_3} \), if \( R \) and \( T \) are saturated (A BA-relation is saturated if it is equal to a Cartesian product of three interval relations), the fundamental operations of intersection, composition and converse between \( R \) and \( T \) are as follows[11]:

**Intersection**

\[
R \cap T = (R \downarrow x \cap T \downarrow x) \times (R \downarrow y \cap T \downarrow y) \times (R \downarrow z \cap T \downarrow z)
\]

**Composition**

\[
R \circ T = (R \downarrow x \circ T \downarrow x) \times (R \downarrow y \circ T \downarrow y) \times (R \downarrow z \circ T \downarrow z)
\]

**Converse**

\[
R^c = R \downarrow x \times R \downarrow y \times R \downarrow z
\]

FROM BCD-RELATIONS TO BA-RELATIONS AND BACK

In this section, we consider the connection between BCD-relations and BA-relations and the translation of them. For better readability, from now on an arbitrary basic BCD-relation will be denoted with the letter ‘\( c \)’, while ‘\( C \)’ will denote an arbitrary relation in \( 2^{B_{bcd}} \). Similarly, ‘\( r \)’, ‘\( R \)’ will denote BA-relations. Sometimes we use ‘\( D \)’ elsewhere to refer to a disjunctive relation of any type.

**From BCD-relation to BA-relation**

For any basic block cardinal direction relation \( C \in B_{bcd} \), there exists only a BA-relation \( R \) such that \( \text{dir}(P, O) = C \) holds if \( P \circ O \) holds. The mapping between \( B_{bcd} \) to \( B_3 \) was given in [5]. For convenience, we call it the Mapping Table (MT for short). The 216 basic block cardinal directions are derived from the Cartesian product of the 6 sets of convex intervals in \( I_A \), namely, \( \text{Idir} = \{\{b, m\}, \{o, fi\}, \{s, d, e, f\}, \{di\}, \{si, oi\}, \{mi, bi\}\} \).

For the set \( \text{Idir} \), let \( k_1 = \{s, d, e, f\} \), \( k_2 = \{m, b\} \), \( k_3 = \{mi, bi\} \), \( k_4 = \{fi, o\} \), \( k_5 = \{si, oi\} \) and \( k_6 = \{di\} \).

**Definition 5.** (Translation of BCD-relations) Let \( c \) and \( D \) be basic and disjunctive BCD-relations, respectively. The function \( TA: 2^{B_{bcd}} \to 2^{B_3} \) is defined recursively as:

- case of basic relations: \( TA(c) = \{r \in B_3 | \text{dir}(P, O) = c \Rightarrow P \circ O\} \),
- case of disjunctive relations: \( TA(D) = \bigcup_{c \in D} TA(c) \).

We can get the MT by using the function \( TA \) applied to \( B_{bcd} \).
Example 3. Consider a block cardinal direction relation \( C=\{\text{Su:SWu}, \text{Sm:SWm}\} \). In reference to the MT and considering the definition of \( TA \), we obtain the translation of \( C \) as: 
\[
TA(C) = \{\text{Su:SWu}, \text{Sm:SWm}\} = \{\text{o, fi}\} \times \{b, m\} \times \{\text{bi, mi}\} \cup \{\text{o, fi}\} \times \{b, m\} \times \{s, d, e, f\} = k4 \times k2 \times k1 , k3 .
\]

From BA-relation to BCD-relation

In this subsection, we define a mapping, called cardinal closure (TC), to translate BA-relations to BCD-relations. In this case the translation will be with loss of information, since the function \( TA \) is non-invertible. This is due to the image of \( TA \) is a proper subset of \( 2^{B3} \). For instance, block algebra \( \{(\text{o, m, bi}, \text{o, mi, b}\} \) is not the image under \( TA \) of any BCD-relation because this algebra corresponds to the relation \( \{\text{Su:SWu, Nd:NWd}\} \), which is not in \( Bcd \).

Definition 6. (Translation of BA-relations) Let \( r \) and \( D \) be basic and disjunctive BA-relations, respectively. The cardinal closure \( TC : 2^{B3} \rightarrow 2^{Bbcd} \) is defined as:

- case of basic relations: \( TC(r) = \{c \in Bbcd \mid r \in TA(c)\} \),
- case of disjunctive relations: \( TC(D) = \bigcup_{r \in D} TC(r) \).

Example 4. In Example 2, the block algebra between \( P \) and \( O \) is \( \{(\text{o, m, b}, \text{o, mi, b}\} \), which corresponds to the Cartesian product of three convex relations in \( Idir \), i.e. \( \{\text{o, fi}\} \times \{b, m\} \times \{b, m\} \). In reference to the MT, we get \( S:SW \) and \( u \), so we have \( dir(P, O) = \text{Su:SWu} \).

One possible application of the function \( TC \) concerns the computation of the algebraic operations of the BCD calculus from the operations of the Block Algebra.

Proposition 1. For all \( C, C1, C2 \in 2^{Bbcd} \),
\[
\begin{align*}
C1 \cap C2 &= TC(TA(C1) \cap TA(C2)) \\
C1 \circ w C2 &= TC(TA(C1) \circ TA(C2)) \\
wcon(C) &= TC(TA(C)) \uparrow
\end{align*}
\]

Proof. BCD-relation \( C1 \in 2^{Bbcd} \) and BA-relation \( TA(C1) \in 2^{B3} \) are semantically equivalent. It means that they are interpreted as the same binary relation over the domain of blocks, or what is the same: \( P C1O \Leftrightarrow PTA(C1)O \), for any blocks \( P \) and \( O \). Similarly, \( C1 \cap C2 \) and \( TA(C1 \cap C2) \) are also semantically equivalent, so \( TC(TA(C1 \cap C2)) = TC(TA(C1) \cap TA(C2)) = TC(C1 \cap C2) = C1 \cap C2 \). Similarly, \( C1 \) and \( TA(C1) \) are semantically equivalent, and so as \( C2 \) and \( TA(C2) \). We get that \( C1 \circ C2 \) and \( TA(C1) \circ TA(C2) \) are also semantically equivalent. From the definition of weak composition, \( C1 \circ w C2 \) is the strongest BCD-relation entailed by \( C1 \circ C2 \), and by definition of \( TC \), \( TC(C1 \circ C2) \) is also the strongest BCD-relation entailed by \( C1 \circ C2 \), then it must be \( C1 \circ w C2 = TC(C1 \circ C2) = TC(TA(C1) \circ TA(C2)) \). By an analogous argument we can affirm that \( wcon(C) = TC(TA(C)) \uparrow \).
The weak composition and weak converse tables for BCD calculus can be obtained as indicated in the above proposition.

THE TRACTABILITY OF CONVEX SUBALGEBRA OF THE BCD CALCULUS

All the convex relations in $2^{B_3}$ are closed under the operations of intersection, converse and composition [12]. We show that the set of convex block cardinal direction relations of the BCD calculus is tractable by using this property.

Definition 7. (Convex BCD-relation) A block cardinal direction $C$ is a convex relation if $TA(C)$ is a convex BA-relation.

Example 5. We say that the BCD-relation $C=\{Su:SWu, SWm, Sm, Su:SWu:Sm:SWm\}$ is convex. In reference to the MT, we have $TA(C)=TA(Su:SWu)\cup TA(SWm)\cup TA(Sm)\cup TA(Su:SWu:Sm:SWm)=[b, f] \times [b, m] \times [si, bi]$, which is a convex BA-relation since it is the Cartesian product of three convex IA relations, where $[b, f]$ is equal to the interval between $b$ and $f$ in the interval lattice[10], in other words, $[b, f]=[b, m, o, fi, d, e, s, f]$. The same works for $[b, m]$ and $[si, bi]$. Thus, by definition, $C$ is a convex BCD-relation.

We get Id by grouping 13 atomic relations in $B_1$. Since any convex BA-relation is the Cartesian product of three convex IA relations, we can obtain the set of convex BCD-relation as follows. First, we choose those convex relations of IA that can be obtained by union of relations from the set Id. We obtain 20 convex relation, denoted by $C_{dir}$, i.e. $C_{dir}=\{\{s, d, e, f\}, \{b, m\}, \{mi, bi\}, \{o, fi\}, \{si, oi\}, \{di\}, \{si, oi, mi, bi\}, \{s, d, e, f, si, oi\}, \{b, m, o, fi, si, oi, mi, bi\}\}$. Second, Navarrete et al. have obtained 400 rectangle algebra relations by $C_{dir}$[8]. In 3D space, we can get $20 \times 20 \times 20=8000$ convex possible BA-relations that correspond to a Cartesian product of three relations from $C_{dir}$. Let $C_{ba}$ be the set of convex BA-relations obtained in this way, which are the only convex BA-relations that belong to the image of $TA$. Finally, we compute the cardinal closure of the relations in $C_{ba}$, generating in this way the set $C_{bcd}$ of 8000 non-empty convex BCD-relations. For any convex BA-relation in $C_{ba}$ w.r.t. a relation in $C_{bcd}$, we have $rx, ry, rz \in C_{dir}$.

Consider the convex relation corresponding to the Cartesian product of three relations from $C_{dir}$. We define a minimum IA relation of an interval.

Definition 8. We name $TD$ a function, for any $T \in 2^{B_1}$, obtains the smallest IA relation that contains $T$ and is belonged to $C_{dir}$.

Proposition 2. For any BA-relation $R \in 2^{B_3}$, we have $TC(R\downarrow x \times R\downarrow y \times R\downarrow z)=TC(TD(R\downarrow x) \times TD(R\downarrow y) \times TD(R\downarrow z))$.

Proof. By definition of $TD$, we have $R\downarrow x \in TD(R\downarrow x)$, $R\downarrow y \in TD(R\downarrow y)$,
Proposition 3. If $R$ is a convex BA-relation then $TC(R)$ is a convex BCD-relation.

Proof. $R$ is a convex BA-relation, so $R = R \downarrow x \times R \downarrow y \times R \downarrow z$. By definition of $TD$, we have $TD(R \downarrow x)$, $TD(R \downarrow y)$, and $TD(R \downarrow z) \in Cdir$. From Proposition 2, we have $TC(R) = TC(TD(R \downarrow x) \times TD(R \downarrow y) \times TD(R \downarrow z))$. Thus, $TC(R)$ is convex since $TD(R \downarrow x) \times TD(R \downarrow y) \times TD(R \downarrow z)$ is convex.

Next we will define the operations of weak composition ($\circ w$) and weak converse ($w\text{con}$) over convex IA-relations from the set $Cdir$.

Definition 9. Let $T$, $T_1$, $T_2 \in Cdir$, we define:

$T_1 \circ_w T_2 := TD(T_1 \circ T_2)$

$w\text{con}(T) := TD(T)$.

The following proposition shows a method for computing the algebraic operations of the convex subalgebra of the BCD calculus using the weak operations on interval relations defined above.

Proposition 4. Let $C$, $C_1$, $C_2$ be the convex BCD-relations. Then

$$C_1 \cap C_2 = TC((TA(C_1) \downarrow x \cap TA(C_2) \downarrow x) \times (TA(C_1) \downarrow y \cap TA(C_2) \downarrow y) \times (TA(C_1) \downarrow z \cap TA(C_2) \downarrow z)) \quad (4)$$

$$C_1 \circ_w C_2 = TC((TA(C_1) \downarrow x \circ TA(C_2) \downarrow x) \times (TA(C_1) \downarrow y \circ TA(C_2) \downarrow y) \times (TA(C_1) \downarrow z \circ TA(C_2) \downarrow z)) \quad (5)$$

$$w\text{con}(C) = TC(w\text{con}(TA(C_1) \downarrow x \times w\text{con}(TA(C_1) \downarrow y \times w\text{con}(TA(C_1) \downarrow z))) \quad (6)$$

Proof. We prove part (5). By proposition 1 and 2, we have $C_1 \circ_w C_2 = TC(TA(C_1) \circ TA(C_2))$.

Proposition 5. The set $Cdir$ is closed under intersection, weak converse and weak composition.

Proof. Suppose $T_1$, $T_2 \in Cdir$. It is easy to see that $T_1 \cap T_2 \in Cdir$. $T_1$, $T_2$ are convex relations of IA, then $T_1 \circ T_2$ and $T_1 \text{areconvex},$ since the set of convex IA-relations is closed under these operations. Hence, $TD(T_1 \circ T_2)$ and $TD(T_1)$ are convex, by Definition 8. Thus $T_1 \circ T_2$ and $T_1 \in Cdir$ by Definition 9. Hence, the proposition holds.
Like in IA, convexity results in tractability. The set $\text{Cba}$ of convex BA-relations, together with the operations of intersection, composition and converse, is a sub algebra of BA [12]. A similar result holds for the set $\text{Cbcd}$ of convex relations of the BCD calculus.

**Theorem 1.** The set $\text{Cbcd}$ is closed under intersection, weak composition and converse, thus, $(\text{Cbcd}, \cap, \text{wcon}, \circ_w)$ is a sub algebra of the underlying algebra of the BCD-calculus, namely, the convex sub algebra.

Proof. Suppose $C, C_1, C_2 \in \text{Cbcd} \subseteq 2^{\text{Bcd}}$. By Proposition 1, $C_1 \cap C_2 = \text{TC}(\text{TA}(C_1) \cap \text{TA}(C_2))$, $C_1 \circ_w C_2 = \text{TC}(\text{TA}(C_1) \circ \text{TA}(C_2))$, $\text{wcon}(C) = \text{TC}(\text{TA}(C))$. By Definition 7, $\text{TA}(C_1)$, $\text{TA}(C_2)$ and $\text{TA}(C)$ are convex BA-relations. Hence $\text{TA}(C_1) \cap \text{TA}(C_2)$, $\text{TA}(C_1) \circ \text{TA}(C_2)$ and $\text{TA}(C)$ are convex BA-relations. Because $\text{Cba}$ is closed under intersection, composition and converse, the cardinal closure $\text{TC}$ is applied to a convex BA-relation in each case. By Proposition 3, $C_1 \cap C_2$, $C_1 \circ_w C_2$ and $\text{wcon}(C)$ are convex. Hence $\text{Cbcd}$ is closed under intersection, weak composition and converse and as a consequence, $(\text{Cbcd}, \cap, \text{wcon}, \circ_w)$ is a sub algebra of the BCD calculus.

As we already know, the convex subset of the BA is tractable[12]. We now exploit the connection between the BA and BCD-calculus to show that the consistency and minimality problem of BCD network can be solved in polynomial time. In other words, the convex sub algebra of BCD calculus is tractable.

A reasoning problem in a qualitative calculus can be formulated as an infinite CSPs (Constraint Satisfaction Problems)$(\mathcal{V}, \mathcal{D}, \Theta)$, where $\mathcal{V}$ is a set of variables interpreted over an infinite domain $\mathcal{D}$ (unlike a finite CSP instance where $\mathcal{D}$ is finite) containing objects of interest, and $\Theta$ is a set of constraints between variables from $\mathcal{V}$ describing qualitative spatial or temporal configurations. A qualitative network can be considered a special infinite CSP instance which can be represented by a complete constraint-labeled digraph $\mathcal{N}=(\mathcal{V}, \mathcal{A}\Theta)$. For the reasoning problem of this paper, the pair of nodes $v_i, v_j \in \Gamma$, the arc from $v_i$ to $v_j$ represents the binary relations between the pair of variables $(v_i, v_j)$, denoted by $C_{ij}$, as well as the constraint formula $v_i C_{ij} v_j$. A relation (constraint) $C_{ij}$ in $\Theta$ is satisfied by an instantiation $\iota$ if the pair $((v_i), \iota(v_j))$ belongs to the binary relation over $\mathcal{D}$ represented by $C_{ij}$. A consistent instantiation or solution of $\Theta$ is an assignment of domain value to variables satisfying all the constraints in $\Theta$. If such a solution exists, we say that the network is consistent, otherwise it is inconsistent. The qualitative network is the BCD-relation or BA-relation network in this paper. In other words, $\mathcal{V}$ represents the blocks in 3D space and $\Theta$ are the BCD-relations or BA-relations.

Balbiani et al. studied the consistency problem for the convex n-block $(n \geq 1$, what we studied is three-dimensional space, where $n = 3$) networks [11]. For saturated BA-networks, they proved the following:
Proposition 6. Let \( \mathcal{N} \) be a convex network, if \( \mathcal{N} \) is path-consistent then: \( \mathcal{N} \) is a consistent network and is minimal.

Based on the connection between BA-relation and BCD-relation, we can solve the consistent problem of Cbcd network by translating it to BA-relation network.

Definition 10. (Translation of BCD-networks) The translation mapping \( TA: 2^{B_{bcd}} \rightarrow 2^{B_3} \) can be extended to translate BCD-networks, in such a way that, for a BCD-network \( \mathcal{N} = (\mathcal{V}, \mathcal{A}, \Theta) \), \( TA(\mathcal{N}) \) can be computed as \( TA(\mathcal{N}) := (TA(\mathcal{V}), TA(\mathcal{A}, \Theta)) \), where \( TR(\mathcal{V}) = \mathcal{V} \) and \( TA(\mathcal{A}, \Theta) \) is obtained by replacing each relation \( C_{ij} \) in \( \mathcal{A}, \Theta \), constraining the pair of variables \( (v_i, v_j) \), by \( R_{ij} = TA(C_{ij}) \cap (TA(C_{ji})) \).

It is important to note that the constraint \( v_i C_{ij} v_j \) cannot be translated directly to \( v_i TA(C_{ij}) v_j \) during the translation, since the network \( TA(\mathcal{N}) \) should meet the condition \( R_{ij} = R_{ji} \) for any pair of variables \( (v_i, v_j) \), which is a precondition when a reasoning algorithm is applied to a qualitative network. For a BCD-network, however, this precondition cannot be imposed, because the BCD-relations are not closed under converse. Be noted that the condition \( C_{ij} = wcon(C_{ij}) \) does not always hold either, since the weak converse of a basic relation may be the disjunction of several BCD-relations. So, we replace \( C_{ij} \) in \( \mathcal{A}, \Theta \) by \( R_{ij} = TA(C_{ij}) \cap (TA(C_{ji})) \).

Theorem 2. A BCD-network \( \mathcal{N} = (\mathcal{V}, \mathcal{A}, \Theta) \) is consistent, if only if \( TA(\mathcal{N}) \) is consistent.

Proof. If a BCD-network \( \mathcal{N} = (\mathcal{V}, \mathcal{A}, \Theta) \) is consistent, then there must exists an instantiation \( \tau \) that assigns blocks to variables so that all the constraints in \( \mathcal{A}, \Theta \) are satisfied. The assignment defines a basis consistent subnetwork \( \mathcal{N}_c \) of \( \mathcal{N} \), in which, for any pair of vairables \( (v_i, v_j) \), the constraints \( v_i C_{ij} v_j \) are satisfied by the instantiation, for some basic relations \( C_{ij} \in C_{ij} \), \( C_{ji} \in C_{ji} \). As a consequence, the constraint \( v_i TA(C_{i j}) \cap TA(C_{j i}) v_j \) must be satisfied by \( \tau \) and \( v_i TA(C_{i j}) \cap TA(C_{j i}) v_j \) is not empty, for any pair of variables \( (v_i, v_j) \), since pair wise consistency is a necessary condition for a consistent network. So \( TA(\mathcal{N}) \) is consistent. Conversely, if \( TA(\mathcal{N}) \) is consistent then there is a consistent scenario in which, for any pair of variables \( (v_i, v_j) \), the constraint \( v_i R_{ij} v_j \) is satisfied for some basis relation \( R_{ij} \in v_i TA(C_{ij}) \cap TA(C_{ji}) v_j \), and so \( v_i C_{ij} v_j \) is also satisfied. Hence, \( \mathcal{N} \) is consistent.

As a result, we get the algorithm CON-Cbcd for checking the consistency of Cbcd network.

**Algorithm: CON-Cbcd**

**Input:** A Cbcd network \( \mathcal{N}_c = (\mathcal{V}, \mathcal{A}, \Theta) \), where \( |\mathcal{V}| = n \),

**Output:** ‘consistent’ if can be satisfied; ‘inconsistent’, otherwise.

**Step:**
1. \( TA(\mathcal{N}_c) := (TA(\mathcal{V}), TA(\mathcal{A}, \Theta)) \) \( \leftarrow \mathcal{N}_c = (\mathcal{V}, \mathcal{A}, \Theta) \) \( \) / \( \) Translation \( \mathcal{N}_c \) to \( TA(\mathcal{N}_c) \).
2. if \( TA(\mathcal{N}_c) \) is empty return ‘inconsistent’;
3. \( \mathcal{N}_a^m \leftarrow QPC(TA(\mathcal{N}_c)) \) / \( \) Apply QPC for \( TA(\mathcal{N}_c) \) to enforce path.
consistency.

4  if \( N^m_a \) is empty then return ‘inconsistent’; //\( N_c \) is inconsistency.
5  else
6       \( N^m_c = TC(N^m_a) \); //Translation \( N^m_c \) to BCD-network.
7  return ‘consistent’;

Analysis of Algorithm: Let \( N_c \) be a convex Cbcd network with \( n \) variables. Qualitative network \( N_c \) is a complete constraint-labeled digraph, which contains \( n^2 - n \) constraints. Algorithm translates the Cbcd network \( N_c \) to convex BA-network \( TA(N_c) \) by definition 10 in step 1, which requires \( O(n^2) \) times to check the M.T. Balbiani et al. proved the following theorem: the consistency problem of a convex block network can be solved in polynomial time by means of path consistency algorithm [12]. Van Beek and Cohen proposed a qualitative path-consistency algorithm for the Interval Algorithm [13], to which we refer as QPC here. The QPC can be re-used in our method, by simply modifying the computation of the algebraic operation. For any three variables \( v_i, v_k \) and \( v_j \), the triangle operation is applied successively, until a stable network is reached: \( R_{ij} = R_{ij} \cap R_{ik} \circ R_{kj} \). In step 3, QPC is used to enforce path consistency of convex BA-network \( TA(N_c) \), which requires \( O(n^3) \). If QPC returns an empty network in step 4 then \( N_c \) is inconsistent. A consistent BA-network is obtained in step 5, which is minimum, since BA-network is saturated (Proposition 6). The process of translating the minimal BA-network \( N^m_a \) to Cbcd network \( N^m_c \) is \( O(n^2) \) in step 6 by applying function TC, due to the number of relations. Algorithm returns ‘consistent’ in step 7, which means that \( N_c \) is a consistent network. Hence, the overall complexity of algorithm is \( O(n^3) \).

Based on the algorithm CON-Cbcd, we have the following theorem:

Theorem 3. Let \( N_c = (V, A \Theta) \) be a Cbcd network, then the consistency and minimality of \( N_c \) be solved in \( O(n^3) \) time.

CONCLUSION

In this paper we consider the connection between BCD-relations and BA-relations. The convex block cardinal direction relations can be translated to the convex block algebra by applying the tractability of the convex block algebra. The tractable subset of convex cardinal direction relations is identified by the fundamental operations of intersection, weak converse and composition of the disjunctive convex block cardinal direction relations. We show that the consistency problem of the qualitative of the convex block cardinal direction relations can be solved in polynomial time by applying path consistency algorithm.

For future work, the problem of identifying larger or maximal tractable subset of BCD calculus is a working direction. Once this problem is solved, the result is helpful in reasoning with cardinal direction in 3D space.
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