The Initial Boundary Value Problem of a Parabolic System

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Abstract. In this paper, the existence and uniqueness of local solutions to the initial and boundary value problem of the system

\[ u_{i t} - \text{div}(|\nabla u|^p \nabla u_i) = f_i(x, t, u_1, u_2, \ldots, u_n), (x, t) \in \Omega_T, \]

are studied. The regularization method is used to construct a sequence of approximation solutions, with the help of monotone iteration technique, then we get the existence of solution of solution of a regularized system. By the use of a standard limiting process, the existence of the local solutions to the system is obtained.

Introduction

The objective of the paper is to study the existence and uniqueness of local solutions to the initial and boundary value problem of the parabolic system

\[ u_{i t} - \text{div}(|\nabla u|^p \nabla u_i) = f_i(x, t, u_1, u_2, \ldots, u_n), (x, t) \in \Omega_T, \tag{1.1} \]

\[ u_i(x, 0) = u_{i0}(x), x \in \Omega, \tag{1.2} \]

\[ u_i(x, t) = 0, (x, t) \in \partial\Omega \times (0, T), \tag{1.3} \]

where \( p_i > 2, i = 1, 2, \ldots, n \), \( \Omega_T = \Omega \times (0, T) \), \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial\Omega \). The conditions of \( f_i \) and \( u_{i0} \) will be given later.

System (1.1) models such as non-Newtonian fluids [1,6] and nonlinear filtration [5], etc. In the non-Newtonian fluids theory, \( p_i (i = 1, 2, \ldots, n) \) are all characteristic quantity of the medium. Media with \( p_i > 2 \) are called dilatant fluids and those with \( p_i < 2 \) are called pseudoplastics. If \( p_i = 2 \), they are Newtonian fluids.

Some authors have studied the global finiteness of the solutions [3,4] and blow up properties of the solutions [9] with various boundary conditions to the equations and systems of evolutionary Laplacian equations. Zhao [11] and Wei-Gao [10] studied the existence and blowing-up property of the solutions to a single equation and the systems of two equations. We found that the method of [10] can be extended to the general systems of \( n \) equations. In this paper, we consider some special cases by stating some method of regularization to construct a sequence of approximation solutions with the help of monotone iteration technique and hence obtain the existence of solutions to a regularized system of equations. Then we obtain the existence of solutions to the system (1.1)-(1.3) by a standard limiting process. Systems (1.1) degenerates when \( u_i = 0 \) or \( \nabla u_i = 0 \). In general, there would be no classical solutions and hence we have to study the generalized solutions to the problem (1.1)-(1.3). The definition of generalized solutions in this work is the following.

Definition 1.1 Function \( u = (u_1, u_2, \ldots, u_n) \) is called a generalized solution of the systems (1.1)-(1.3) if \( u_i \in L^r(\Omega_T) \cap L^\infty(0, T; W_{0}^{1,p}(\Omega)) \), \( u_{i0} \in L^r(\Omega_T), i = 1, 2, \ldots, n \), and satisfies
\[
\int_{\Omega_T} (-u_x \varphi_x + |\nabla u|^{p-2} \nabla u \nabla \varphi) \, dx \, dt - \int_{\Omega} (u_{0\varphi}(x,0)) \, dx = \int_{\Omega_T} f_i(x,t,u_1,u_2,\ldots,u_n) \varphi \, dx \, dt,
\]
\[i \in 1, 2, \ldots, n.
\]

for any \( \varphi \in C^1(\overline{\Omega_T}) \), \( \varphi(x,T) = 0, \varphi(x,t) = 0, (x,t) \in \partial\Omega \times (0,T), i = 1, 2, \ldots, n \).

Equations (1.4) implies that
\[
\int_0^t \int_\Omega (-u_x \varphi_x + |\nabla u|^{p-2} \nabla u \nabla \varphi) \, dx \, dt + \int_\Omega u_i(x,t) \varphi(x,t) \, dx - \int_\Omega u_{0i} \varphi(x,0) \, dx
\]
\[= \int_0^t \int_\Omega f_i(x,t,u_1,u_2,\ldots,u_n) \varphi \, dx \, dt, \quad \text{a.e. } t \in (0,T).
\]

The following are the constraints to the nonlinear functions \( f_i \), \( i = 1, 2, \ldots, n \), involved in this paper.

**Definition 1.2** A function \( f = f(u_1, u_2, \ldots, u_n) \) is said to be quasimonotone nondecreasing (resp., nonincreasing) if for fixed \( u_j (j \neq i) \), \( f \) is nondecreasing (resp., nonincreasing) in \( u_i \) \( (i = 1, 2, \ldots, n) \).

Our main existence result is following:

**Theorem 1.3** If there exist nonegative functions \( f_i(x,t,u_1,u_2,\ldots,u_n) \in C(\overline{\Omega} \times [0,T] \times \mathbb{R}^n) \), which are quasimonotonically nondecreasing for \( u_1, u_2, \ldots, u_n \), and a nonnegative function \( g(s) \in C^1(\mathbb{R}) \) such that
\[
|f_i(x,t,u_1,u_2,\ldots,u_n)| \leq \min\{g(u_i), g(u_2), \ldots, g(u_n)\}, \quad u_{0i} \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega).
\]
\( (i = 1, 2, \ldots, n) \). Then there exists a constant \( T' \in [0,T] \) such that (1.1)-(1.3) has a solution \( u = (u_1, u_2, \ldots, u_n) \) in the sense of Definition 1.1 with \( T \) replaced by \( T' \).

**Theorem 1.4** Assume the \( f = (f_1, f_2, \ldots, f_n) \) is Lipschitz continuous in \( (u_1, u_2, \ldots, u_n) \), then the solution of (1.1)-(1.3) is unique.

The proof of Th1.4 is simple, so we omit it.

**Proof of Theorem 1.3**

To prove the theorem, we consider the following regularized problem
\[
u_{it} - div(|\nabla u|^{p-2} \nabla u) = f_{ix}(x,t,u_1,u_2,\ldots,u_n), (x,t) \in \Omega_T,
\]
\[u_{i0} = 0, (x,t) \in \partial\Omega \times (0,T),
\]
\[u_{i0} = u_{0i}(x), x \in \Omega,
\]
\[u_{i0} = 0, (x,t) \in \partial\Omega \times (0,T),
\]
where \( f_{ix} \in C^1(\overline{\Omega} \times [0,T] \times \mathbb{R}^n) \), \( f_{ix} \) is quasimonotone nondecreasing and \( f_{ix} \to f_i \) uniformly on bounded subsets of \( \overline{\Omega} \times [0,T] \times \mathbb{R}^n \), also
\[
|f_{ix}(x,t,u_1,u_2,\ldots,u_n)| \leq \min\{g(u_i), g(u_2), \ldots, g(u_n)\},
\]
\[u_{0i} \in C^1(\Omega), |u_{0i}|_{L^\infty} \leq \sup_{t \in [0,T]} |u_{0i}|_{L^\infty}, |\nabla u_{0i}|_{L^p} \leq C |\nabla u_{0i}|_{L^p}, u_{0i} \to u_i \text{ strongly in } W_0^{1,p}(\Omega).
\]

**Lemma 2.1** The regularized problem (2.1)-(2.3) has a generalized solution.

**Proof:** Starting from a suitable initial iteration \((u_{i0}^{(0)}, u_{20}^{(0)}, \ldots, u_{n0}^{(0)})\), we construct a sequence \(\{(u_{i0}^{(k)}, u_{20}^{(k)}, \ldots, u_{n0}^{(k)})\}\) from the iteration process
\[
u_{it}^{(k)} - div(|\nabla u_{it}^{(k)}|^{p-2} \nabla u_{it}^{(k)}) = f_{ix}(x,t,u_{10}^{(k-1)}, u_{20}^{(k-1)}, \ldots, u_{n0}^{(k-1)}), (x,t) \in \Omega_T,
\]
\[u_{i0}^{(k)} = 0, (x,t) \in \partial\Omega \times (0,T),
\]
\[u_{i0}^{(k)} = u_{0i}(x), x \in \Omega,
\]
\[u_{i0}^{(k)} = 0, (x,t) \in \partial\Omega \times (0,T),
\]
\begin{align}
\tag{2.5}
u^{(k)}_{\varepsilon}(x,0) &= u_{0\varepsilon}(x), x \in \Omega,
\end{align}

\begin{align}
\tag{2.6}
u^{(k)}_{\varepsilon}(x,t) &= 0, (x,t) \in \partial \Omega \times (0,T),
\end{align}

where \( i = 1, 2, \ldots, n \). It is clear that for each \( k = 1, 2, \ldots \), the above system consists of \( n \) nondegenerated and uncoupled initial boundary-value problems.

By classical results (see, [8]) for fixed \( \varepsilon \) and \( k \), the problem (2.4)-(2.6) has a classical solution \((u^{(k)}_{\varepsilon}, u^{(k)}_{\varepsilon}, \ldots, u^{(k)}_{\varepsilon})\) if \((u^{(k-1)}_{\varepsilon}, u^{(k-1)}_{\varepsilon}, \ldots, u^{(k-1)}_{\varepsilon})\) is smooth.

To ensure that this sequence converges to a solution of (2.4)-(2.6), it is necessary to choose a suitable initial iteration. The choice of this function depends on the type of quasimonotone property of \((f_1, f_2, \ldots, f_n)\). In the following, we establish the monotone property of the sequence.

Set \( \bar{u}^{(0)}_{\varepsilon}(x,t) = \sup_{\Omega} \{u_{0\varepsilon}(x)\}, i = 1, 2, \ldots, n \). Let \( \bar{u}^{(1)}_{\varepsilon} \) be a classical solution of the following problem.

\begin{align}
\tag{2.7}
u^{(1)}_{\varepsilon} - \text{div}\left(\left|\nabla u^{(1)}_{\varepsilon}\right|^2 + \varepsilon \left|\nabla u^{(1)}_{\varepsilon}\right|^2 \right) = f^{(1)}_{\varepsilon}(x,t, u^{(0)}_{\varepsilon}, u^{(0)}_{\varepsilon}, \ldots, u^{(0)}_{\varepsilon}), (x,t) \in \Omega_T,
\end{align}

\begin{align}
\tag{2.8}
\bar{u}^{(0)}_{\varepsilon}(x,0) &= u_{0\varepsilon}, x \in \Omega,
\bar{u}^{(1)}_{\varepsilon}(x,t) &= 0, (x,t) \in \partial \Omega \times (0,T),
\end{align}

By induction, we may obtain a nondecreasing sequence of smooth functions

\begin{align}
\tag{2.7}
u^{(k)}_{\varepsilon} \leq \bar{u}^{(k)}_{\varepsilon} \leq \bar{u}^{(k-1)}_{\varepsilon} \cdots \leq \bar{u}^{(2)}_{\varepsilon} \leq \bar{u}^{(1)}_{\varepsilon} \geq \bar{u}^{(0)}_{\varepsilon}.
\end{align}

In a similar way, by setting \( \underline{u}^{(0)}_{\varepsilon}(x,t) = \inf_{\Omega} \{u_{0\varepsilon}(x)\}, i = 1, 2, \ldots, n \), we can get a solution \((\underline{u}^{(1)}_{\varepsilon}, \underline{u}^{(1)}_{\varepsilon}, \ldots, \underline{u}^{(1)}_{\varepsilon})\) of

\begin{align}
\tag{2.8}
\underline{u}^{(k)}_{\varepsilon} - \text{div}\left(\left|\nabla u^{(k)}_{\varepsilon}\right|^2 + \varepsilon \left|\nabla u^{(k)}_{\varepsilon}\right|^2 \right) = f^{(k)}_{\varepsilon}(x,t, u^{(0)}_{\varepsilon}, u^{(0)}_{\varepsilon}, \ldots, u^{(0)}_{\varepsilon}), (x,t) \in \Omega_T,
\end{align}

\begin{align}
\tag{2.8}
\underline{u}^{(0)}_{\varepsilon}(x,0) &\geq u_{0\varepsilon}, x \in \Omega,
\underline{u}^{(1)}_{\varepsilon}(x,t) &\geq 0, (x,t) \in \partial \Omega \times (0,T),
\end{align}

with

\begin{align}
\tag{2.8}
\underline{u}^{(k)}_{\varepsilon} \leq u^{(k)}_{\varepsilon} \leq u^{(k)}_{\varepsilon} \leq \cdots \leq u^{(2)}_{\varepsilon} \leq u^{(1)}_{\varepsilon} \leq u^{(0)}_{\varepsilon}.
\end{align}

In the same way as above, we obtain a nonincreasing sequence of smooth functions

\begin{align}
\tag{2.8}
\underline{u}^{(0)}_{\varepsilon} \geq \underline{u}^{(1)}_{\varepsilon} \geq \underline{u}^{(2)}_{\varepsilon} \cdots \geq \underline{u}^{(k)}_{\varepsilon} \geq \cdots
\end{align}

It is obvious that \( \underline{u}^{(0)}_{\varepsilon} \leq \bar{u}^{(0)}_{\varepsilon} \). By induction, we assume that \( \underline{u}^{(k)}_{\varepsilon} \leq \bar{u}^{(k)}_{\varepsilon} \). Since \( f_{\varepsilon} \) is quasimonotone nondecreasing, we have

\begin{align}
\tag{2.8}
f^{(k)}_{\varepsilon}(x,t, u^{(0)}_{\varepsilon}, u^{(0)}_{\varepsilon}, \ldots, u^{(0)}_{\varepsilon}) \leq f^{(k)}_{\varepsilon}(x,t, u^{(k)}_{\varepsilon}, u^{(k)}_{\varepsilon}, \ldots, u^{(k)}_{\varepsilon}) \leq f^{(k)}_{\varepsilon}(x,t, u^{(k)}_{\varepsilon}, u^{(k)}_{\varepsilon}, \ldots, u^{(k)}_{\varepsilon}) \leq f^{(k)}_{\varepsilon}(x,t, u^{(k)}_{\varepsilon}, u^{(k-1)}_{\varepsilon}, \ldots, u^{(1)}_{\varepsilon}) \leq f^{(k)}_{\varepsilon}(x,t, u^{(k)}_{\varepsilon}, u^{(k-1)}_{\varepsilon}, \ldots, u^{(1)}_{\varepsilon}) \leq f^{(k)}_{\varepsilon}(x,t, u^{(k)}_{\varepsilon}, u^{(k-1)}_{\varepsilon}, \ldots, u^{(1)}_{\varepsilon})
\end{align}
for $i = 1, 2, \ldots, n$. 

$$
\text{div}(\nabla u_{ie}^{(k+1)}) + \varepsilon \frac{p-2}{2} \nabla u_{ie}^{(k+1)} = f_{ie}(x,t,u_{ie}^{(k)},u_{2e}^{(k)}, \ldots, u_{ne}^{(k)}), (x,t) \in \Omega_T,
$$

$$
\text{div}(\nabla u_{ie}^{(k)}) - \varepsilon \frac{p-2}{2} \nabla u_{ie}^{(k)} = f_{ie}(x,t,u_{ie}^{(k)},u_{2e}^{(k)}, \ldots, u_{ne}^{(k)}), (x,t) \in \Omega_T,
$$

$$
u_{ie}^{(k+1)}(x,0) = u_{0e}(x,0), x \in \Omega,
$$

$$
u_{ie}^{(k+1)}(x,t) = 0 = \nu_{ie}^{(k)}(x,t), (x,t) \in \partial \Omega \times (0,T),
$$

By the comparison principle, we have $u_{ie}^{(k)} \leq u_{ie}^{(k+1)}$. Therefore

$$
\lim_{k \to \infty} u_{ie}^{(k)} = u_{ie}, \text{ a.e. in } \Omega_T
$$

(2.10)

By the continuity of $f_{ie}(i=1,2,\ldots,n)$, we have

$$
\lim_{k \to \infty} f_{ie}(x,t,u_{ie}^{(k)},u_{2e}^{(k)}, \ldots, u_{ne}^{(k)}) = f_{ie}(x,t,u_{ie},u_{2e}, \ldots, u_{ne}) \quad \text{a.e. in } \Omega_T
$$

(2.11)

We now prove that there exist a $T' \in (0,T]$ and a constant $M$ (independent of $k$ and $\varepsilon$) such that for all $k$, we have

$$
\left| u_{ie}^{(k)} \right|_{C^1(\Omega_T)} \leq M, \quad i = 1, 2, \ldots, n.
$$

(2.12)

Let $v_i^+(t)$ be the solutions of the ordinary differential equations

$$
\frac{dv_i^+}{dt} = g(v_i), \quad v_i^+(0) = \pm \left| u_{0i} \right|_{C^1(\Omega)}, (i = 1, 2, \ldots, n).
$$

By standard results in [2], there exist $T_i \in (0,T), i = 1, 2, \ldots, n$, such that $v_i^+$ exists on $[0,T_i]$ with $T_i$ depends only on $\left| u_{0i} \right|_{C^1(\Omega)}$. By the comparison theorem

$$
\left| u_{ie}^{(k)}(x,t) \right| \leq \max \{v_i^+(t), v_i^-(t)\}, \quad (i = 1, 2, \ldots, n).
$$

Setting $T' = \frac{1}{n} \min\{T_1, T_2, \ldots, T_n\}, M = \max\{v_i^+(T'), -v_i^-(T')\}$, we obtain (2.12).

We now claim that $u_{ie}^{(k)} \rightharpoonup u_{ie}$, as $k \to \infty$, weakly in $L^p(0,T;W^{1,p}_0(\Omega)) \ (i = 1, 2, \ldots, n)$, where $\rightharpoonup$ stands for weak convergence.

Multiplying (2.4) by $u_{ie}^{(k)}$ and integrating over $\Omega_T$, we obtain that

$$
\left| \nabla u_{ie}^{(k)} \right|_{L^p(\Omega_T)}^p \leq \int_{\Omega_T} \left| \nabla u_{ie}^{(k)} \right|^p + \varepsilon \frac{p-2}{2} \left| \nabla u_{ie}^{(k)} \right|^2 \ dx \, dt \leq C,
$$

(2.13)

where $C$ is a constant independent of $\varepsilon, k$.

Multiplying (2.4) by $u_{ie}^{(k)}$ and integrating over $\Omega_T$, by Cauchy inequality and integrating by parts, we obtain

$$
\int_{\Omega_T} \left| u_{ie}^{(k)} \right|^2 \ dx \, dt \leq C \left\{ \int_{\Omega_T} \left| \nabla u_{ie}^{(k)} \right|^p \ dx \right\} + \frac{1}{\varepsilon} \int_{\Omega_T} f_{ie}^2(x,t,u_{ie}^{(k-1)},u_{2e}^{(k-1)}, \ldots, u_{ne}^{(k-1)}) \ dx \, dt \leq C.
$$

(2.14)
By (2.13) and (2.14), we obtain that there exists a subsequence of \( u_{i\varepsilon}^{(k)} \) converging weakly in following sense as \( j \to \infty \).

\[
\nabla u_{i\varepsilon}^{(k)} \rightharpoonup \nabla u_{i\varepsilon}, \quad \text{weakly in} \quad L^p(\Omega_T),
\]

(2.15)

\[
|\nabla u_{i\varepsilon}^{(k)}|^{p-2} u_{i\varepsilon}^{(k)} \rightharpoonup w_{i\varepsilon}^{(k)}, \quad \text{weakly in} \quad L^{\frac{p}{p-1}}(\Omega_T), \quad \text{for some} \quad w_{i\varepsilon}^{(k)},
\]

(2.16)

\[
u_{i\varepsilon}^{(k)} \rightharpoonup v_{i\varepsilon}, \quad \text{weakly in} \quad L^2(\Omega_T),
\]

(2.17)

Similar as [11], we can prove that \( w_{i\varepsilon} = |\nabla u_{i\varepsilon}|^{p-2} u_{i\varepsilon}, \) \( i = 1, 2, \ldots, n \).

Following (2.10), (2.11), (2.13) and (2.14) and the uniqueness of the weak limits, it is easy to know that, as \( k \to \infty \),

\[
\nabla u_{i\varepsilon}^{(k)} \rightharpoonup \nabla u_{i\varepsilon}, \quad \text{weakly in} \quad L^p(\Omega_T),
\]

(2.18)

\[
u_{i\varepsilon}^{(k)} \to v_{i\varepsilon}, \quad f_{i\varepsilon}(x,t,u_{i\varepsilon}^{(k)},u_{i\varepsilon}',u_{i\varepsilon}^{(k)}) \to f_{i\varepsilon}(x,t,u_{i\varepsilon},u_{i\varepsilon}',u_{i\varepsilon}'), \quad \text{a.e. in} \quad \Omega_T,
\]

(2.19)

\[
u_{i\varepsilon}^{(k)} \rightharpoonup v_{i\varepsilon}, \quad \text{weakly in} \quad L^2(\Omega_T).
\]

(2.20)

\[
|\nabla u_{i\varepsilon}^{(k)}|^{p-2} u_{i\varepsilon}^{(k)} \to |\nabla u_{i\varepsilon}|^{p-2} u_{i\varepsilon}, \quad \text{weakly in} \quad L^{\frac{p}{p-1}}(\Omega_T), \quad i = 1, 2, \ldots, n.
\]

(2.21)

Combining the above results, we have proved that \( u_{i\varepsilon} = (u_{i\varepsilon},u_{i\varepsilon}',u_{i\varepsilon}^{(k)}), \quad (x,t) \in \Omega_T \) is a generalized solution of (2.1)-(2.3).

**Proof of theorem 1.3**

Since \( u_{i\varepsilon} \) satisfy similar estimates as (2.12)-(2.14), combining the property of \( f_{i\varepsilon} \), we know that there are functions \( u_{i\varepsilon} \in L^p(0,T;W^{1,p}_0(\Omega)) (i = 1,2,\ldots,n) \) as \( \varepsilon \to 0 \).

By a standard limiting process, \( (u_{i\varepsilon},u_{i\varepsilon}',u_{i\varepsilon}^{(k)}) \to (u_1,u_2,\ldots,u_n) \). Thus \( u = (u_1,u_2,\ldots,u_n) \) is a generalized solution of (1.1)-(1.3).

**References**


