Quantum Isometry Groups of Group with Different Generating Sets

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Abstract. The main goal of this paper is to discuss the structure of the quantum isometry groups associated to the discrete two matrix $Z_2$-group $G(2,Z_2)$ and Dihedral group $D_n$, and then we show that the quantum isometry groups $Q(G(2,Z_2))$ of $G(2,Z_2)$ with two different generating sets are isomorphic to $D_θ \left( C^*(D_θ), \Delta_{D_θ} \right) := C^*(D_θ \oplus D_θ)$, where $θ$ is a automorphism of compact quantum group $Q(G(2,Z_2))$. The quantum isometry group $Q(D_θ)$ of $D_θ$ with the presentation (3) is not isomorphic to $D_θ \left( C^*(D_θ), \Delta_{D_θ} \right)$ except the case: One of $D$, $D^*, B$ and $C$ is zero. But the quantum isometry group $Q(D_θ)$ of with the presentation (4) is isomorphic to $C^*(D_θ \oplus D_θ)$.

Introduction

Symmetry is one of the most fundamental concepts in mathematics and physics. The representation of the group formed by the symmetry of the physical system plays a very important role in the study of the physical system. The symmetry of groups is naturally extended to the symmetry given by actions of quantum groups on the algebra of observables. Since the formulation of quantum automorphism groups by S. Wang [1], following suggestions of Alain Connes, many interesting examples of such quantum groups, particularly the quantum permutation groups of finite sets, finite graphs and quantum isometry groups, have been extensively studied by a number of mathematicians (see, e.g.[2-12]

In [13,14], Soltan et al defined the doubling procedure and shown that, the quantum isometry group of group $C^* - algebra$ is isomorphic to finite quantum group obtained as doubling of the group algebra with standard cocommutative Hopf algebra structure.

We believe that a detailed study of finite groups will not only give many new and interesting results on compact quantum groups, but also contribute to the understanding of the structure of finite quantum group. For this, it is important to discuss the structure of the quantum isometry groups associated to the discrete two matrix $Z_2$-group $G(2,Z_2)$ and $D_θ$. By using the construction method of quantum isometry group in paper [4,6,9], we further investigate the quantum isometry groups of $G(2,Z_2)$ and $D_θ$ with different generating sets. The rest of the paper is organized as follows: In Section 2, we introduce some of basic notations and terminologies of the quantum automorphism group. In section 3, we give some results of the quantum isometry groups of $G(2,Z_2)$ and $D_θ$ with different generating sets.

Preliminaries

Bhowmick and Goswami gave the following definition of quantum isometry groups of spectral triples in [6].

Definition 2.1 Let $(A^∞, H, D)$ be a spectral triple of compact type (a la Connes) and $Q(D)=Q(A^∞, H, D)$ be a category whose objects are $(Q, U)$, where $(Q, Δ)$ is a compact quantum group having a unitary representation $U$ on the Hilbert space $H$ such that:

1. $Δ$ commutes with $D \otimes 1_Q$.
2. $(1D \otimes θ) \circ ad_D(a) \in A^∞$ for all $a \in A^∞$ and $θ$ is any state on $Q$, where $ad_D(a):= θ(a \otimes 1)θ^*$ for $a \in B(H)$.

A CQG morphism between two objects $(Q_1, U_1)$ and $(Q_2, U_2)$ is a compact quantum group morphism $π: Q_1 \rightarrow Q_2$ such that $U_2 = (id \otimes π)U_1$. If a universal object exists in $Q(D)$, then we denote it
by $QISO^+(A^{\infty}, H, D)$ and the corresponding largest Woronowicz subalgebra is called the quantum group of orientation preserving isometries and denote by $QISO^+(A^{\infty}, H, D)$.

Authors [6] show that $QISO^+(A^{\infty}, H, D)$ exists if $(A^{\infty}, H, D)$ be a spectral triple of compact type, and $D$ has some good properties.

In [4], the authors provides a more direct way of computing $QISO^+(A^{\infty}, H, D)$, by using the following definition and theorem:

**Definition 2.3** Let $\mathcal{C}(A^{\infty}, H, D)$ be the category with objects $(Q, \alpha)$, where $Q$ is a compact quantum group and $\alpha$ is a $C^*$-action of $Q$ on $C^*$-closure $A$ of $A^{\infty}$ inside $B(H)$ such that:

1. $\alpha$ is $\tau$ preserving, i.e., $\left(\alpha \otimes \text{id}\right)(a) = \tau(a) I$ for all $a \in A$.
2. $\alpha$ maps $A_{\text{clos}}$ into $A_{\text{clos}} \otimes Q$.
3. $\alpha D = (\tilde{D} \otimes I) \alpha$.

The morphism in $\mathcal{C}(A^{\infty}, H, D)$ are compact quantum group morphisms intertwining the respective actions.

**Theorem 2.4** [4, 6]. There exists a universal object $\tilde{Q}$ in $\mathcal{C}(A^{\infty}, H, D)$. It is isomorphic to $QISO^+(A^{\infty}, H, D)$.

Let $\Gamma$ be a finitely generated discrete group with generating set $S = \{a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_k, a_k^{-1}\}$.

An integer valued function $l$ on a group $(\Gamma, S)$ is a length function on $\Gamma$ by $d(g, h) = l(gh^{-1})$, then $d$ is a left $\Gamma$-invariant metric on $\Gamma$.

Define the operator $D_\Gamma$ on $l^2(\Gamma)$ by $\text{Dom}(D_\Gamma) = \{\eta \in l^2(\Gamma) : \sum_{g \in \Gamma} l(g)^2 |\eta_g|^2 < \infty\}$,

$D_\Gamma (\eta)(g) = l(g) \eta(g)$. $\eta \in \text{Dom}(D_\Gamma)$. If $g$ belongs to $\Gamma$, we will denote by $\delta_g$ the function in $l^2(\Gamma)$ which takes the value $1$ at the point $g$ and $0$ at all other points. It is well known that $(C(\Gamma), l^2(\Gamma), D_\Gamma)$ is a spectral triple.

**Remark 2.5** Let $(C(\Gamma), l^2(\Gamma), D_\Gamma)$ denote the spectral triple as above. In [4], authors have shown that $QISO^+(C(\Gamma), l^2(\Gamma), D_\Gamma)$ exists and depend on the generating set. We will denote it by $Q(\Gamma, S)$. According to Theorem 2.4, (for detail See [4,6,11]), $Q(\Gamma, S)$ is also the universal object in the category $\mathcal{C}(C(\Gamma), l^2(\Gamma), D_\Gamma)$.

Representation of the Group $G(2, Z_2)$ and Quantum Isometry Groups

**Representation of the Group $G(2, Z_2)$ and Its Computation of Quantum Isometry Group**

We know that $Z_2 = \{0, 1\}$ is finite domain with the following operations:

$+: 0 + 0 = 0, 0 + 1 = 1, 1 + 0 = 1, 1 + 1 = 0$;

$\cdot: 0 \cdot 0 = 0, 0 \cdot 1 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$.

The $Z_2$-m X n-matrix is defined:

$a = (a_{ij})_{m \times n}, b = (b_{ij})_{m \times n}, c = (c_{ij})_{m \times n}$, where $m$ represents $m$ rows, $n$ represents $n$ columns. Let $a = (a_{ij})_{m \times n}, b = (b_{ij})_{m \times n}, c = (c_{ij})_{m \times n}$, when $k = n$, then we define the matrix multiplication:

$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$.

Let’s define a $Z_2$-matrix group in the following form:

$G(2, Z_2) = \{a = (a_{ij})_{2 \times 2} | a_{ij} \in Z_2, i = 1, 2, j = 1, 2, a_{11} \cdot a_{22} + a_{21} \cdot a_{12} = 1\}$.

Obviously, $G(2, Z_2)$ is a group of six elements.

We shall let

$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The standard generating set for the $D_2$ group $G(2, Z_2)$ is $S = \{a, b\}$ with $a^2 = e$ and $b^3 = e$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Further, we obtain the following representation theorem.

**Theorem 3.1**. The discrete $Z_2$ –matrix group of $G(2, Z_2)$ is a Dihedral group of order 6 with the following two presentations:
\[(G(2,\mathbb{Z}_2), S^2) = \langle a, b | a^2 = b^3 = e, ab = b^{-1}a \rangle. \quad (1)\]

\[(G(2,\mathbb{Z}_2), S^2) = \langle a, c | a^2 = (ac)^2 = e, \rangle. \quad (2)\]

By using [11, Theorem 5.1], [4, Theorem 4.9], we have the following description of the quantum isometry group.

**Theorem 3.2** The quantum isometry groups \(Q(G(2,\mathbb{Z}_2))\) of the discrete \(\mathbb{Z}_2\)-matrix group of order 2 isomorphic to \(D^g_8 (C^*(D_8), \Delta_{D^g_8}) \cong C^*(D^e_8 \oplus D^g_8)\), where \(\theta\) is a automorphism of compact quantum group \(Q(G(2,\mathbb{Z}_2))\) with \(\theta^2 = 1\) and the automorphism \(\theta\) is given by \(\theta a = b, \theta b = a\), \(D^g_8 (C^*(D_8), \Delta_{D^g_8})\) is a doubling of \((C^*(D_8), \Delta)\) corresponding to a given automorphism \(\theta\).

**Computation of Quantum Isometry Group for \(G(2,\mathbb{Z}_2)\)**

From Theorem 3.2, we know that different generative sets of \(G(2,\mathbb{Z}_2)\), we get the same quantum isometry groups \(Q(G(2,\mathbb{Z}_2))\). It is well known that Dihedral group \(D_8\) of order 8 has two different generating sets with the following two presentations:

\[(D_8, S^2) = \langle a, b | a^2 = b^4 = e, ab = b^{-1}a \rangle. \quad (3)\]

\[(D_8, S^2) = \langle a, c | a^2 = (ac)^4 = e, \rangle. \quad (4)\]

According to [11, Theorem 5.1], we know that the quantum isometry groups \(Q(D_8, S^2)\) of \((D_8, S^2)\) with presentation (4) is isomorphic to \(D^g_8 (C^*(D_8), \Delta_{D^g_8}) \cong C^*(D^e_8 \oplus D^g_8)\), where \(\theta\) is a automorphism of compact quantum group \(Q(D_8, S^2)\) with \(\theta^2 = 1\).

But when \(D_8\) with the presentation (3), we can’t be sure that \(Q(D_8, S^2)\) is isomorphic to \(C^*(D^e_8 \oplus D^g_8)\). In order to compare the two cases of (3) and (4). Hence by using the method in [6], we further discuss the properties of the quantum isometry groups \(Q(D_8, S^2)\) of \((D_8, S^2)\) with presentation (3).

Let \(\alpha\) be the action of quantum isometry group \(Q(D_8, S^2)\) on \(C^*(D_8)\) by the following form:

\[\alpha(\lambda_a) = \lambda_a \otimes A + \lambda_b \otimes B + \lambda^{-1}_{b-1} \otimes C\]

\[\alpha(\lambda_b) = \lambda_a \otimes D + \lambda_b \otimes E + \lambda^{-1}_{b-1} \otimes F\]

In order to obtain the characterization of quantum isometry group \(Q(D_8, S^2)\) of \((D_8, S^2)\) with presentation (3), we first give the following Lemmas:

**Lemma 3.3.**

\[A^2 + BC + CB = I, \quad AB + CA = 0, \quad AC + BA = 0, \quad B^2 + C^2 = 0.\]

**Proof.** By \(a^2 = e, a = a^{-1}\), we have,

\[\alpha(\lambda_a) = \alpha(\lambda_a) \alpha(\lambda_a) = \lambda_a \otimes I = \lambda_a \otimes (A^2 + BC + CB) + \lambda_{b^2} \otimes B^2 + \lambda_{b^{-1}} \otimes C^2\]

According to \(b^4 = e, ab = b^{-1}a\), then \(b^2 = b^{-2} = ba = ab^{-1}\),

\[\lambda_a \otimes I = \lambda_a \otimes (A^2 + BC + CB) + \lambda_{b^2} \otimes (B^2 + C^2)\]

\[+ \lambda_{ab} \otimes (AB + CA) + \lambda_{b_{a^{-1}}} \otimes (BA + AC)\]

\[\alpha(\lambda_a) = \alpha(\lambda_{a^{-1}}) = \lambda_a \otimes A^* + \lambda_b \otimes C^* + \lambda_{b^{-1}} \otimes B^*\]
By comparing the coefficients of \( \lambda_\alpha, \lambda_\beta, \lambda_{ab} \) and \( \lambda_{ba} \) in \( \alpha(\lambda_\alpha) = \alpha(\lambda_\beta) \), the coefficients of \( \lambda_\alpha, \lambda_\beta \) and \( \lambda_{ba-1} \) in \( \alpha(\lambda_\alpha) = \alpha(\lambda_\beta) \), we have above results.

**Lemma 3.4.**

\[
DD^* + EE^* + FF^* = I, \quad DF^* + FD^* = 0, \quad DE^* + ED^* = 0, \quad EF^* + FE^* = 0.
\]

**Proof.** Since \( bb^{-1} = e \), we have,

\[
\alpha(\lambda_\alpha) = \alpha(\lambda_\beta) \alpha(\lambda_{ba-1}) = \lambda_\alpha \otimes I = \lambda_\alpha \otimes (DD^* + EE^* + FF^*) + \lambda_{ab} \otimes EF^* + \lambda_{ba-1} \otimes FE^* + \lambda_{ab} \otimes DF^* + \lambda_{ba} \otimes DE^* + \lambda_{ba} \otimes FD^* + \lambda_{ba} \otimes (DE^* + ED^*)
\]

According to \( b^4 = e, ab = b^{-1}a, \) then \( b^2 = b^{-2}, b\alpha = ab^{-1} \)

\[
\alpha(\lambda_\alpha) = \lambda_\alpha \otimes (DD^* + EE^* + FF^*) + \lambda_{ab} \otimes EF^* + \lambda_{ba} \otimes FD^* + \lambda_{ba} \otimes (DE^* + ED^*)
\]

By comparing the coefficients of \( \lambda_\alpha, \lambda_\beta, \lambda_{ab} \) and \( \lambda_{ba} \), we have above results.

**Lemma 3.5.**

\[
AD + BF + CE = D^*A + E^*B + F^*C, \quad AE + CD = D^*B + E^*A,
\]

\[
AF + BD = D^*C + F^*A, \quad BE + CF = F^*B + E^*C.
\]

**Proof.** According to \( \alpha(\lambda_\alpha) = \lambda_\alpha \otimes A + \lambda_\beta \otimes B + \lambda_{ba} \otimes C \)

\[
\alpha(\lambda_\beta) = \lambda_\beta \otimes D + \lambda_{ab} \otimes E + \lambda_{ba} \otimes F
\]

\[
\alpha(\lambda_{ba-1}) = \lambda_{ba} \otimes D^* + \lambda_{ba} \otimes F^* + \lambda_{ba} \otimes E^*.
\]

(7)

By comparing the coefficients of \( \lambda_\alpha, \lambda_\beta, \lambda_{ab} \) and \( \lambda_{ba} \) in \( \alpha(\lambda_{ba-1}) = \alpha(\lambda_{ab}) \), we have above results.

**Theorem 3.6** One of \( D, \ D^*, B \) and \( C \) is zero, and the other three are zero. In this case, we have

\[
A^2 = AA^* = I, \quad EE^* + FF^* = I, \quad F = AF^*A, \quad E = AE^*A.
\]

**Proof.** When \( D = 0 \), then \( D^* = 0 \). Applying the antipode on the relations (5) and (6), we have, \( B = 0, C = 0 \). According to Lemma 3.3, Lemma 3.4, we know that \( A^2 + BC + CB = I, \quad DD^* + EE^* + FF^* = I, \) then, we have \( A^2 = AA^* = I, \quad EE^* + FF^* = I. \) By Lemma 3.5, we have \( AE = E^*A \) in \( AE + CD = D^*B + E^*A, \) \( AF = F^*A \) in \( AF + BD = D^*C + F^*A \). Hence, \( F = AF^*A, \quad E = AE^*A \) by \( A^2 = AA^* = I. \) For other cases, we use similar methods to get the same result.

**Theorem 3.7** If one of \( D, \ D^*, B \) and \( C \) is zero, then the quantum isometry groups \( Q(D_b, S^1) \) of \( D_b \) with the presentation (3) isomorphic to \( C^* (D_b \oplus D_b) \) as \( C^* - algebra \). Its action \( \alpha \) on \( C^* (D_b) \) is given by the formula:

\[
\alpha(\lambda_\alpha) = \lambda_\alpha \otimes A,
\]

\[
\alpha(\lambda_\beta) = \lambda_\beta \otimes E + \lambda_{ba} \otimes F
\]

\[
\alpha(\lambda_{ba-1}) = \lambda_{ba} \otimes F^* + \lambda_{ba} \otimes E^*.
\]

And the coefficients of the quantum isometry groups \( Q(D_b, S^1) \) have the following relations: \( A^2 = AA^* = I, EE^* + FF^* = I, F = AF^*A, E = AE^*A. \)

**Proof.** According to Theorem 3.6, we have, One of \( D, \ D^*, B \) and \( C \) is zero, and the other three are zero. Hence, \( D, B \) and \( C \) is zero in (5) and (6). \( D^* \) is zero in (7). From Theorem 3.6. We have, \( A^2 = AA^* = I, EE^* + FF^* = I, F = AF^*A, E = AE^*A. \)
Summary

From Theorem 3.2, Theorem 3.7, [11,Theorem 5.1] and [4, Theorem 4.9], we find that the key of the quantum isometry groups coincides with the doubling of the group algebra is One of D, B and C is zero. We know that the quantum isometry groups Q(G(2,Z_2)) with the action satisfied (5) and (6), then we have D^* is zero by [4, Theorem 4.9]. But the quantum isometry groups Q(\tilde{\mathcal{D}}_B, S^2) with the action satisfied (5) and (6) does not produce D^* is zero.

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References