Asymptotic Behavior of Global Positive Solution to an Information Diffusion Model with Random Perturbation in Social Network

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Abstract. The paper explores an information diffusion model with random perturbation in social network. First, we show the model exit the unique global positive solution. By the construction of the Lyapunov function, we give the positive solution is stochastically asymptotically stable in the large around disease-free equilibrium, i.e. the conditions of the information diffusion will die out, investigate the stochastic asymptotic behavior of the positive solution around endemic equilibrium of the deterministic models, obtain the stochastic asymptotic stability condition, i.e. the conditions of the information diffusion will be persistent in social networks.

Introduction

Social media such as blogs, discussion forums, and social networking sites provide new channels for individuals to share information and express their opinions. Research on social networks has received remarkable attention in the past decade, the diffusion process on the online social network became an ongoing research topic [1-7]. Due to similar patterns in the spread of epidemics and social contagion processes, most research adopts the same theoretical principles for epidemics in describing the information diffusion. Mathematical models are used extensively in the study of information diffusion phenomena.

Jaime Mena-Lorcat, Herbert W. Hethcote [8] investigate the most basic SIRS models is as follows

\[
\begin{align*}
\dot{S}(t) &= \lambda - \beta S(t)I(t) - \mu S(t) + \delta R(t), \\
\dot{I}(t) &= \beta S(t)I(t) - (\mu + \epsilon + \gamma)I(t), \\
\dot{R}(t) &= \gamma I(t) - (\mu + \delta)R(t),
\end{align*}
\]

(1)

where the parameter \(\lambda, \beta, \mu, \epsilon, \gamma, \delta\) are positive constants, and \(S(t), I(t), R(t)\) denote the number of the individuals susceptible to the disease, of infected members and of members who have been removed from the possibility of infection through full immunity, respectively. Some notable features of the model: The influx of individuals into the susceptible is given by a constant \(\lambda\); it is assumed that the natural death rates are assumed to be equal (denoted by constant \(\mu\)) and individuals in \(I(t)\) suffer an additional death due to disease with rate constant \(\epsilon\), \(\beta, \delta\) and \(\gamma\) represent the disease transmission coefficient, the rate of losing their immunity on diseases and the rate of recovery from infection, respectively[8].

For describing the information diffusion in social network, we revise the model (1), let the constant \(\epsilon = 0\), since information can’t lead to death. The positive constants \(\lambda, \beta, \mu, \gamma, \delta\) are the same as the system (1). On the other hand, the real world networks is full of randomness and stochasticity, using stochastic models can gain more real benefits. Consequently, Some researchers have paid their
attention to the stochastic Information Diffusion model [9,10]. We assumed that stochastic
perturbations were of white noise type, consider that the rate of recovery from infection coefficient \( \gamma \) was subject to stochastic perturbations in model (1), i.e. the fluctuations in the environment will manifest themselves mainly as fluctuations in the parameter \( \gamma \),

\[
\gamma \rightarrow \gamma + \sigma B(t),
\]

where \( B(t) \) is standard Brownian motions with \( B(0) = 0 \), and with intensity of white noise \( \sigma^2 \). The equations with random perturbation of information diffusion be deduced:

\[
\begin{align*}
    dS(t) &= ( (\lambda - \beta S(t)I(t) - \mu S(t) + \delta R(t)) \) \) dt, \\
    dI(t) &= ( \beta S(t)I(t) - (\mu + \gamma) I(t)\) \) dt - \sigma I(t) dB(t), \\
    dR(t) &= ( \gamma I(t) - (\mu + \delta) R(t)\) \) dt + \sigma R(t) dB(t).
\end{align*}
\]

Where \( S(t) \) denotes the number of future population who might have an interest in information at time \( t \), \( I(t) \) denotes the number of current population who know information at time \( t \). \( R(t) \) denotes the number of past population who lose interest in information at time \( t \). For convenience, \( S(t), I(t), R(t) \) are still named the susceptible, infectious, and recovered on information diffusion in social network, respectively.

Similar to [8], It is obvious that the deterministic model corresponding to the stochastic equations (2) have a disease-free equilibrium \( E_0 = \left( \frac{\lambda}{\mu}, 0, 0 \right) \), an endemic equilibrium \( E^* = \left( \frac{\mu + \gamma, \beta(\beta \lambda - \mu(\mu + \gamma))}{\beta \mu(\mu + \gamma + \delta)}, \frac{\gamma(\beta \lambda - \mu(\mu + \gamma))}{\beta \mu(\mu + \gamma + \delta)} \right) \), and the basic reproduction number \( R_0 = \frac{\lambda \beta}{\mu(\mu + \gamma)} \). If \( R_0 \leq 1 \), the disease-free equilibrium \( E_0 \) is globally asymptotical stable, then the information diffusion will die out. If \( R_0 > 1 \), the disease-free equilibrium \( E_0 \) becomes unstable and the endemic equilibrium \( E^* \) is global asymptotically stable endemic equilibrium which implies the information diffusion always remains.

This paper is organized as follows. In Section 2, we show there is a unique positive solution of system (3). In Section 3, we deduce the condition \( R_0 \leq 1 \), investigate the asymptotic behavior of the positive solution around disease-free equilibrium the disease-free equilibrium \( E_0 \) of the system (2). In Section 4, we deduce the condition \( R_0 > 1 \), investigate the asymptotic behavior of the positive solution, the condition for the information diffusion being persistent is given.

Next, we give some basic theory in stochastic differential equations. Throughout this paper, unless otherwise specified, let \( \Omega, \{ F_t \}_{t \geq 0}, \mathbb{P} \) be a complete probability space with a filtration \( \{ F_t \}_{t \geq 0} \) satisfying the usual conditions (i.e. it is right continuous and \( F_0 \) contains all \( \mathbb{P} \)-null sets). Let

\[
\mathbb{R}^d_+ = \{ x \in \mathbb{R}^d \mid x_i > 0, \forall i \leq d \}, \quad \mathbb{R}^d_0 = \{ x \in \mathbb{R}^d \mid x_i \geq 0, \forall i \leq d \}.
\]

In general, consider the \( n \)-dimensional stochastic differential equation

\[
\begin{align*}
    dx(t) &= f(x(t),t) dt + g(x(t),t) dB(t)
\end{align*}
\]

with initial value \( x(t_0) = x_0 \in \mathbb{R}^d \). \( B(t) \) denotes \( n \)-dimensional standard Brownian motion defined on the above probability space. Define the differential operator \( L \) associated with (5)

\[
L = \frac{\partial}{\partial t} + \sum_{i=1}^d f_i(x,t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d [g^T(x,t)g(x,t)] \frac{\partial^2}{\partial x_i \partial x_j}.
\]
If $L$ acts on a function $V \in C^2_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}_+^3; \mathbb{R}_+^3)$, then
\[
EV(S_t \wedge T), I_t \wedge T), R(t \wedge T) \leq V(S(0), I(0), R(0)) + KES(t \wedge T),
\]
where $EV(S(t \wedge T), I(t \wedge T), R(t \wedge T) \leq V(S(0), I(0), R(0)) + KT$ and $V_{\xi} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{t \xi}$. By Itô’s formula, if $x(t) \in \mathbb{R}^d$, then
\[
dV(x(t), t) = LV(x(t), t)dt + V_y(x(t), t)g(x(t), t)dB(t).
\]
Consider (4), assume $f(0, t) = 0$ and $g(0, t) = 0$ for all $t \geq t_0$. So $x(t) \equiv 0$ is a solution of (5), called the trivial solution or equilibrium position.

**Existence and Uniqueness of Positive Solution**

In this section we first show that the solution of system (3) is global and positive. To get a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition [11]. However, the coefficients of system (3) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of system (3) may explode in finite time. In this section, using the Lyapunov analysis method, we show the solution of system (1) is positive and global.

**Theorem 1.** There is a unique solution $(S(t), I(t), R(t))$ of system (2) on $t \geq 0$ for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$, and the solution will remain in $\mathbb{R}_+^3$ with probability 1, namely, $(S(t), I(t), R(t)) \in \mathbb{R}_+^3$ for all $t \geq 0$ almost surely.

**Proof.** Since the coefficients of the equation are locally Lipschitz continuous for any given initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$, there is a unique local solution $(S(t), I(t), R(t))$ on $t \in [0, \tau_\varepsilon)$, where $\tau_\varepsilon$ is the explosion time (see [11]). To show this solution is global, we need to show that $\tau_\varepsilon = \infty$ a.s. Let $m_0 > 0$ be sufficiently large so that $S(0) \in \left[\frac{1}{m_0}, m_0\right], I(0) \in \left[\frac{1}{m_0}, m_0\right]$ and $R(0) \in \left[\frac{1}{m_0}, m_0\right].$ For each integer $m > m_0$, define the stopping time $\tau_m = \inf\{t \in [0, \tau_\varepsilon) : S(t) \notin \left[\frac{1}{m}, m\right], I(t) \notin \left[\frac{1}{m}, m\right] \text{ or } R(t) \notin \left[\frac{1}{m}, m\right]\}$, where we set $\inf \emptyset = \infty$ as usual (\emptyset denotes the empty set). According to the definition, $\tau_m$ is increasing as $m \to \infty$. Set $\tau_\varepsilon = \lim_{m \to \infty} \tau_m$, whence $\tau_\varepsilon \leq \tau_m$ a.s. If we can show that $\tau_\varepsilon \leq \infty$ a.s., then $\tau_\varepsilon \leq \tau_m$ a.s., and $(S(t), I(t), R(t)) \in \mathbb{R}_+^3$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_\varepsilon \leq \infty$ a.s. If this statement is false, then there exist a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P(\tau_\varepsilon \leq T) > \varepsilon.$ Hence there is an integer $m_1 \geq m_0$ such that for all $m \geq m_1$ we have
\[
P(\tau_m \leq T) \geq \varepsilon.
\]

Define a $C^2$-function $V : \mathbb{R}_+^3 \to [0, +\infty)$ by $V(S, I, R) = (S - a - a \log \frac{S}{a}) + (I - 1 - \log I) + (R - 1 - \log R),$ where $a > 0$ is a real positive constant to be chosen later. The nonnegativity of this function $V(S, I, R)$ can be seen from $u - a - a \log \frac{u}{a} \geq 0$ for all $u > 0, a > 0$.

Let $m \geq m_1$ and $T > 0$ be arbitrary. Applying the Itô’s formula, we obtain
\[
dV(S, I, R) = (1 - \frac{a}{S})dS + (-1 + \frac{1}{T})dI + \frac{1}{2}dR = (S) = \frac{1}{2}R^2dR + (1 - \frac{1}{R})dR + \frac{1}{2}R^2(dR)^2 = L dt + \sigma(-I + R)dB(t),
\]
where \( LV: \mathbb{R}^3_+ \to [0, +\infty) \) is defined by

\[
LV = (\lambda + a\mu + 2\mu + \gamma + \delta + \sigma^2) - (\mu + \beta) - \frac{a}{S} - a\beta - \frac{1}{R} + (a\beta - \mu)I - \mu R. \tag{8}
\]

Let \( a = \frac{\mu}{\beta} \) such that \( a\beta - \mu = 0 \). Substituting this into (8), we get

\[
LV = (\lambda + a\mu + 2\mu + \gamma + \delta + \sigma^2) - (\mu + \beta) - \frac{a}{S} - a\beta - \frac{1}{R} - \frac{\gamma}{R} = K.
\]

Substituting this into (7), we get \( dV \leq Kdt + \sigma(-I + R)dB(t) \). Integrating this from 0 to \( \tau_n \wedge T \) and dividing \( t \) on both sides and taking the expectation, we have

\[
EV(S(\tau_n \wedge T), I(\tau_n \wedge T), R(\tau_n \wedge T)) \leq V(S(0), I(0), R(0)) + KT \tag{9}
\]

Let \( \Omega_m = \{\tau_n \wedge T\} \), for \( m \geq m \), and by (6), we have \( P(\Omega_m) \geq \epsilon \). Note that for every \( \omega \in \Omega_m \), there is at least one of \( S(\tau_n, \omega), I(\tau_n, \omega) \) and \( R(\tau_n, \omega) \) that equals either \( m \) or \( \frac{1}{m} \), and hence

\[
V(S(\tau_n \wedge \omega), I(\tau_n \wedge \omega), R(\tau_n \wedge \omega)) \geq \min\{m - 1 - \log m, \frac{1}{m} - 1 + \log m, m - a - a\log m, \frac{1}{m} - a - a\log \frac{1}{mk}\}.
\]

It then follows from (9) that

\[
V(S(0), I(0), R(0)) + KT \geq E[1_{\Omega_m}(\omega)V(S(\tau_n \wedge \omega), I(\tau_n \wedge \omega), R(\tau_n \wedge \omega))]
\]

\[
\geq \epsilon \min\{m - 1 - \log m, \frac{1}{m} - 1 + \log m, m - a - a\log m, \frac{1}{m} - a - a\log \frac{1}{mk}\},
\]

where \( 1_{\Omega_m} \) is the indicator function of \( \Omega_m \). Letting \( m \to \infty \), leads to the contradiction

\[
\infty > V(S(0), I(0), R(0)) + KT = \infty.
\]

So we must therefore have \( \tau_n = \infty \). Namely, \( (S(t), I(t), R(t)) \in \mathbb{R}^3_+ \) for all \( t \geq 0 \) almost surely.

Then, for the system (3) has unique solution \( (S(t), I(t), R(t)) \) on \( t \geq 0 \) for any initial value \( (S(0), I(0), R(0)) \in \mathbb{R}^3_+ \), and the solution will remain in \( \mathbb{R}^3_+ \) with probability 1. This finishes the proof.

**Asymptotic Behavior Around \( E_0 \)**

If \( R_0 \leq 1 \), then \( E_0 \) is globally stable, which means the information will die out after some period of time. Hence, it is interesting to study the disease-free equilibrium for controlling information diffusion. In this section, we show the average oscillation around \( E_0 \) in time to exhibit whether the information will die out.

**Theorem 2.** If \( R_0 = \frac{\beta \lambda}{\mu(\mu + \gamma)} \leq 1 \) and the condition is satisfied \( \sigma^2 < \min\{2\mu, \frac{2\mu(\mu + \delta)}{2\mu + \gamma}\} \), then the solution \( (S(t), I(t), R(t)) \) of system (3) with initial value \( (S(0), I(0), R(0)) \in \mathbb{R}^3_+ \) is stochastically asymptotically stable in the large.

**Proof.** Let \( u = S - \frac{\lambda}{\mu}, v = I \), and \( w = R \) then \( u \in R, v > 0, w > 0 \) and \( du(t) = (-\mu u - \beta u v - \beta^2 v + \delta w)dt, \)

\[
dv(t) = [(\mu + \gamma - \beta^2) v]dt - \sigma_v dB(t), \quad dw(t) = [\gamma v - (\mu + \delta) w]dt + \sigma_w dB(t).
\]

Define a \( C^2 \) function \( V: \mathbb{R}^3_+ \to [0, +\infty) \) by \( V(u, v, w) = (u + v + w)^2 + c_1 v + c_2 w^2 \), where \( c_1 > 0 \) and \( c_2 > 0 \) are real positive constants to be chosen later. The function \( V(u, v, w) \) is positive-definite.
Using Itô’s formula, we compute
\[ dV = LVdt - \sigma((2(u + v + w) + c_1)v + 2(u + v + (1 + c_1)w)w)dB(t), \]
where
\[ LV = 2(u + v + w)(-\mu u - \mu v - \mu w) + (v^2 + w^2)\sigma^2 + c_1[\beta uv - (\mu + \gamma - 2\mu)w] + 2c_2w[\gamma v - (\mu + \delta)w] + c_2\sigma^2w^2 \]
\[ = -2\mu(u^2 - v^2) - 2[\mu + c_1(\mu + \delta)]w^2 + (c_1\beta - 4\mu)uv + 2(c_2\gamma - 2\mu)vw - 4\mu uw - c_1\beta \frac{\lambda}{\mu}\sigma^2(1 - 1)v + \sigma^2(v^2 + (1 + c_1)w^2). \]

We choose \( c_1 = \frac{4\mu}{\beta} \) and \( c_2 = \frac{2\mu}{\gamma} \) such that \( c_1\beta - 4\mu = 0 \) and \( c_2\gamma - 2\mu = 0 \). We can obtain
\[ LV = -2\mu u^2 - 2\mu v^2 - \frac{2[\mu\gamma + 2\mu(\mu + \delta)]}{\gamma}w^2 - 4\mu uw - c_1\beta \frac{\lambda}{\mu}\sigma^2(1 - 1)v + \sigma^2v^2 + \frac{2\mu + \gamma}{\gamma}\sigma^2w^2. \]

From \( R_0 \leq 1 \) and \( v > 0 \), we have \(-c_1\beta \frac{\lambda}{\mu}\sigma^2(1 - 1)v \leq 0\). Moreover, using Cauchy inequality such that
\[ 2\frac{\lambda}{\mu} \leq u^2 + \frac{\alpha^2}{\mu^2} \text{ and } -2uv \leq \frac{u^2}{\rho} + \frac{v^2}{\rho}, \text{ set } \rho = \frac{\gamma + \mu + \delta}{\gamma}, \text{ hence,} \]

\[ LV = -\frac{2\mu(\mu + \delta)}{\gamma + \mu + \delta}u^2 - 2\mu v^2 - \frac{2\mu(\mu + \delta)}{\gamma}w^2 + \sigma^2v^2 + \frac{2\mu + \gamma}{\gamma}\sigma^2w^2 \]
\[ = -\frac{2\mu(\mu + \delta)}{\gamma + \mu + \delta}u^2 - (2\mu - \sigma^2)v^2 - \left(\frac{2\mu(\mu + \delta)}{\gamma} - \frac{2\mu + \gamma}{\gamma}\sigma^2\right)w^2 \leq 0. \]

Consequently, \( LV \) is negative-definite. This combined with the definite of \( V \) gives that the solution \((S(t), I(t), R(t))\) of system (3) with initial value \((S(0), I(0), R(0)) \in \mathbb{R}_+^3\) is stochastically asymptotically stable in the large. This finishes the proof of Theorem 2.

### Asymptotic Behavior Around \( E^* \)

In this section, we show the average oscillation around \( E^* \) in time to exhibit whether the information will be persistent.

**Theorem 3.** If \( R_0 = \frac{\beta\lambda}{\mu(\mu + \gamma)} > 1 \), and the condition is satisfied \( \sigma^2 < \min\{2\mu, \frac{2\mu(\mu + \delta)}{2\mu + \gamma}\} \), then the solution \((S(t), I(t), R(t))\) of system (2) with initial value \((S(0), I(0), R(0)) \in \mathbb{R}_+^3\) has the property
\[ \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left[(S(r) - S^*)^2 + (I(r) - I^*)^2 + (R(r) - R^*)^2\right] dr \leq \frac{K_\sigma}{M}. \]

Where \( E^* = (S^*, I^*, R^*) \) is the endemic equilibrium of system (1), and
\[ M = \min\left\{ \frac{\mu(\mu + \delta)}{\gamma + \mu + \delta} - \frac{1}{2}\sigma^2, \frac{2\mu(\mu + \delta) - (2\mu + \gamma)\sigma^2}{2\gamma}\right\}, \]
\[ K_\sigma = \frac{\mu\sigma^2}{2\mu - \sigma^2}I^* + \frac{2\mu(\mu + \delta)(2\mu + \gamma)\sigma^2}{2\gamma(2\mu(\mu + \delta) - (2\mu + \gamma)\sigma^2)}R^* + \frac{\mu}{\beta}I^*\sigma^2. \]

**Proof.** Define a \( C^2 \) function \( V : \mathbb{R}_+^3 \to [0, +\infty) \) by

...
\[ V(x) = \frac{1}{2} (S - S^+ + I - I^+ + R - R^-)^2 + a(I - I^- - I^+ \log \frac{I}{I^+}) + \frac{1}{2} p(R - R^-), \]
where \(a > 0, p > 0\) are real positive constants to be chosen later.

Applying the Itô’s formula, we have
\[
dV(x) = LVdt + [a(I - I^-) - \sigma I(S - S^+ + I - I^+ + R - R^-)]dB(t) + \sigma R[S - S^+ + I - I^+ + (1 + p)(R - R^-)]dB(t) \quad (11)
\]
where
\[
LV = (S - S^+ + I - I^+ + R - R^-)(\lambda - \mu S - \mu R) + \frac{1}{2} \sigma^2 (I^2 + R^2) + a(I - I^-)[\beta S I - (\mu + \gamma)I] \\
+ \frac{1}{2} a I^2 \sigma^2 + p(R - R^-)[\gamma I - (\mu + \delta)R] + \frac{1}{2} p \sigma^2 R^2 \\
= -\mu(S - S^+)^2 - \mu(I - I^+)^2 - \mu + p(\mu + \delta)(R - R^-)^2 + (\alpha \beta - 2 \mu)(S - S^+)(I - I^+) \\
+ (\gamma - 2 \mu)(I - I^+)(R - R^-) - 2 \mu(S - S^+)(R - R^-) + \frac{1}{2} \sigma^2 (I^2 + R^2) + \frac{1}{2} a I^2 \sigma^2 + \frac{1}{2} p \sigma^2 R^2
\]

Let \(a = \frac{2 \mu}{\beta}\) such that \(\alpha \beta - 2 \mu = 0\), choose \(p = \frac{2 \mu}{\gamma}\) such that \(p \gamma - 2 \mu = 0\). This implies
\[
LV = -\mu(S - S^+)^2 - \mu(I - I^+)^2 - \mu + p(\mu + \delta)(R - R^-)^2 \\
-2 \mu(S - S^+)(R - R^-) + \frac{1}{2} \sigma^2 (I^2 + R^2) + \frac{2 \mu}{\beta} I^2 \sigma^2 + \frac{2 \mu}{\gamma} p \sigma^2 R^2
\]

Set \(\rho = \frac{\gamma + \mu + \delta}{\gamma}\), such that \(-2(S - S^+)(R - R^-) \leq \frac{\gamma}{\gamma + \mu + \delta}(S - S^+)^2 + \frac{\gamma + \mu + \delta}{\gamma}(R - R^-)^2\), then we compute
\[
LV \leq \frac{\mu(\mu + \delta)}{\gamma + \mu + \delta}(S - S^+)^2 - \mu(I - I^+)^2 - \mu + \frac{2 \mu}{\beta} I^2 \sigma^2 - \mu + \frac{\mu(\mu + \delta)}{\gamma}(R - R^-)^2 + \frac{1}{2} \sigma^2 (I^2 + R^2) + \frac{\mu}{\gamma} \sigma^2 R^2 \\
= -\frac{\mu(\mu + \delta)}{\gamma + \mu + \delta}(S - S^+)^2 - (\mu - \frac{1}{2} \sigma^2)(I - \frac{2 \mu}{\mu - \sigma^2} I^+)^2 - \frac{2 \mu(\mu + \delta)}{\gamma}(R - R^-)^2 + \frac{1}{2} \sigma^2 (I^2 + R^2) + \frac{\mu}{\gamma} \sigma^2 R^2
\]

From the condition \(\sigma^2 < \min\{2 \mu, \frac{2 \mu(\mu + \delta)}{2 \mu + \gamma}\}\), we have \(\mu - \frac{1}{2} \sigma^2 > 0\) and \(\frac{2 \mu(\mu + \delta) - (2 \mu + \gamma) \sigma^2}{2 \gamma} > 0\), then we have
\[
LV \leq \frac{\mu(\mu + \delta)}{\gamma + \mu + \delta}(S - S^+)^2 - \mu(I - I^+)^2 - \mu + \frac{2 \mu}{\beta} I^2 \sigma^2 - \mu + \frac{\mu(\mu + \delta)}{\gamma}(R - R^-)^2 + \frac{1}{2} \sigma^2 (I^2 + R^2) + \frac{\mu}{\gamma} \sigma^2 R^2 \\
= -\frac{\mu(\mu + \delta)}{\gamma + \mu + \delta}(S - S^+)^2 - (\mu - \frac{1}{2} \sigma^2)(I - \frac{2 \mu}{\mu - \sigma^2} I^+)^2 \\
- \frac{2 \mu(\mu + \delta) - (2 \mu + \gamma) \sigma^2}{2 \gamma}(R - \frac{2 \mu(\mu + \delta)}{2 \mu + \gamma} \sigma^2 R^2)^2 + K_\sigma
\]

Substituting this into (11), we get
\[
dV(x) \leq LVdt + [a(I - I^-) - \sigma I(S - S^+ + I - I^+ + R - R^-)]dB(t) + \sigma R[S - S^+ + I - I^+ + (1 + p)(R - R^-)]dB(t)
\]

Integrating the (11) from 0 to \(t\) and taking the expectation, we have
\[
\limsup_{t \to \infty} \frac{1}{t} E \left[ (S(t) - S^+)^2 + (I(t) - \frac{2 \mu}{2 \mu - \sigma^2} I^+)^2 + (R(t) - \frac{2 \mu(\mu + \delta)}{2 \mu + \delta} - (2 \mu + \gamma) \sigma^2 R^2)^2 \right] dt \leq \frac{K_\sigma}{M}
\]

This finishes the proof of Theorem 2.
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References


