A Simple Method for the Calculation of Induced Electromotive Force

Wei CHEN

School of Sciences, Hubei University of Automotive Technology, Shiyan, Hubei, China, 442002
mark@huat.edu.cn

Keywords: Induced Electromotive Force, Magnetic Field, Probability.

Abstract. A new simple method for the calculation of induced electromotive force is introduced, with specified condition of an infinitely long straight wire placed in a uniform symmetry circular magnetic field.

Introduction

In college physics teaching, we are acquainted with the expression of the induced electromotive force: \( \varepsilon = \oint E_k \cdot d\bar{l} \), but \( E_k \) is limited to the case with circular symmetry when calculation, namely, the time-varying uniform magnetic field exciting induced electric field also has circular symmetry. For a finitely long straight wire placed in such a magnetic field, there are two calculation methods for induced electromotive force on the wire: firstly, direct calculation using distribution of the induced electric field and \( \oint E_k \cdot d\bar{l} \); the other one is to set up auxiliary lines to constitute a loop, and calculate the electromotive force of the loop using \( \varepsilon = -\frac{d\phi}{dt} \), with corresponding treatment[1]. For the calculation of an infinitely long straight wire placed in the magnetic field, the former method is usually used. In fact, with proper adjustments, the latter approach can also be used. The following calculation shows an example.

Integration Method

If the uniform magnetic field is limited to a circular area of radius \( R \), and the rate of change versus time \( \frac{dB}{dt} > 0 \), for an infinitely long straight wire placed in the magnetic field, which has chord length \( b \) cut by this round, the string corresponding central angle is \( 2\theta \), and the distance from center of the circle to this chord is \( h \), as shown in figure 1. Considering the symmetry of wire, the induced electromotive force between A and B is

\[
\varepsilon_{AB} = 2\int_{l_1} E_{k1} \cdot d\bar{l} + 2\int_{l_2} E_{k2} \cdot d\bar{l}
\]

(1)
where $\mathbf{E}_1$ and $\mathbf{E}_2$ are respectively the induced electric field strength of the zones of $r < R$ and $r > R$, with areas equal to $\frac{r}{2} \frac{\partial B}{\partial t}$ and $\frac{R^2}{2r} \frac{\partial B}{\partial t}$. It is not difficult to tell that the above-mentioned power line of the induced electric field is a set of concentric circles with this magnetic field. Its direction is counter-clockwise, to substitute $\mathbf{E}_1$ and $\mathbf{E}_2$ into the expression of equation 1, considering the angle integrals between the electric field $\mathbf{E}_k$ and $d\mathbf{l}$, equation 1 becomes

$$\varepsilon_{AB} = 2 \int_1^r \frac{r}{2} \frac{\partial B}{\partial t} \, dl \cos a + 2 \int_2^{r} \frac{R^2}{2r} \frac{\partial B}{\partial t} \cos a \, dl$$  \hspace{1cm} (2)$$

We can see from the figure $R$ and $r$ have following relation

$$\frac{h}{r} = \cos a$$  \hspace{1cm} (3)$$

$$l = \text{htga}$$  \hspace{1cm} (4)$$

From equation 4

$$dl = h \sec^2 a \cdot da$$  \hspace{1cm} (5)$$

to substitute equation 3 and 5 into equation 2, and using $\cos a \cdot \sec a = 1$, equation 2 is simplified to:

$$\varepsilon_{AB} = 2 \int_0^{b/2} h \frac{\partial B}{\partial t} \, dl + 2 \int_1^2 \frac{R^2}{2} \frac{\partial B}{\partial t} \, da$$  \hspace{1cm} (6)$$

$$\varepsilon_{AB} = \frac{1}{2} bh \frac{\partial B}{\partial t} + \frac{R^2}{2} (\pi - 2\theta) \frac{\partial B}{\partial t}$$  \hspace{1cm} (7)$$

By using $b/2 = R \sin \theta$, $h = R \cos \theta$, the electromotive force of infinitely long straight wire is

$$\varepsilon_{AB} = \frac{R^2}{2} \frac{\partial B}{\partial t} (\sin 2\theta + \pi - 2\theta)$$  \hspace{1cm} (8)$$
Its direction is from A to point B; electric potential of B point is higher than A point. The above calculation is the usual method. Because of the usage of calculus, we might call it "Integration method".

**Probability Method**

We can also consider from another perspective. Since it is an infinitely long straight wire, it is possible to imagine the two "endpoints" at infinity join each other and constitute a closed loop. By calculating the change rate of the magnetic flux that passes through the loop versus time, the electromotive force in the wire can be found. The rationality of this method will be proved as follows:

We set the relative position of infinitely long straight wire to the original magnetic field unchanged as figure 1 to make a comparison. The infinitely long wire is joint at the infinity and constitutes a closed-loop ABCA. The region 1 of the magnetic field is surrounded by this loop, as shown in Figure 2. If the area of region 1 in figure 2 is $S_1$, from the geometric relationship in the figure we have

\[
S_1 = \frac{1}{2} R^2 (2\theta) - \frac{1}{2} \cdot 2R \sin \theta \cdot R \cos \theta
\]

\[
= R^2 \theta - \frac{1}{2} R^2 \sin 2\theta
\]

(9)

If the clockwise direction is the forward direction of this loop, the direction of the magnetic field is perpendicular to paper and towards paper, with $\frac{\partial \mathbf{B}}{\partial t} > 0$. By Faraday’s law of electromagnetic induction, the electromotive force in the loop is $\varepsilon_1 = -\frac{d(\mathbf{B} \cdot \mathbf{S}_1)}{dt}$.

\[
\varepsilon_1 = -(R^2 \theta - \frac{1}{2} R^2 \sin 2\theta) \frac{\partial \mathbf{B}}{\partial t}
\]

(10)

The negative sign in equation 10 indicates that the real direction of the electromotive force in the loop is opposite to the assumed direction. In other words, the direction of electromotive force in the loop ABCA is counter-clockwise. For the AB segment, its electromotive force is pointed from B to A.

Apparently this infinitely long wire can also be joint at the infinity and constitutes a closed-loop ABDA, the region 2 of the magnetic field surrounded by this loop, as shown in Figure 2. If the area of the region 2 in figure 2 is $S_2$, from the geometric relationship in the figure we have

\[
S_2 = \pi R^2 - S_1
\]

\[
= R^2 (\pi - \theta) + \frac{1}{2} R^2 \sin 2\theta
\]

(11)

If the clockwise direction is still taken as the forward direction of this loop, similarly using Faraday's law of electromagnetic induction, the electromotive force in the loop is
\[ \varepsilon_2 = -\frac{d(\mathbf{B} \cdot \mathbf{S}_z)}{dt} \]

\[ \varepsilon_2 = \frac{1}{2} R^2 (\pi - \theta) + \frac{1}{2} R^2 \sin 2\theta \frac{dB}{dt} \]  \hspace{1cm} (12)

The negative sign in equation 12 indicates that the real direction of the electromotive force in the loop is opposite to the assumed direction. In other words, the direction of electromotive force in the loop ABDA is counter-clockwise. For the AB segment, its electromotive force direction is pointed from A to B.

However, how can we decide to take ABCA loop, or ABDA loop in the calculation? According to the model of the infinitely long straight wire, its two end points may join each other at "infinity" which is below the wire, to constitute loop ABCA. It is also possible that two end points join each other at "infinity" which is above the wire to constitute a loop ABDA. The probability of these two cases are both 1/2. EMF \( \varepsilon_1, \varepsilon_2 \) of Wire AB’s two ends is a set of random variables; its distribution is shown in table 1

<table>
<thead>
<tr>
<th></th>
<th>( \varepsilon )</th>
<th>( \varepsilon_1 )</th>
<th>( \varepsilon_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(\varepsilon) )</td>
<td>( P_1 )</td>
<td>( P_2 )</td>
<td></td>
</tr>
</tbody>
</table>

According to the principle of mathematical statistics, the EMF at both ends of wire AB should be expected as:

\[ \varepsilon_{AB} = p_1 \varepsilon_1 + p_2 \varepsilon_2 \]  \hspace{1cm} (13)

From the above calculation we can see that the actual direction of \( \varepsilon_1 \) is from B to point A, reversed to \( \varepsilon_{AB} \). The actual direction of the \( \varepsilon_2 \) is from A to point B, the same to \( \varepsilon_{AB} \). Therefore, in the calculation \( \varepsilon_1 < 0, \varepsilon_2 > 0 \) should be taken. The probability of the existence of the two loops are 1/2 each. Therefore, \( p_1 = p_2 = 1/2 \), are substituted into equation 13 and taking into account equation 10 and 12, the electromotive force at "unlimited long" straight wire AB both ends is

\[ \varepsilon_{AB} = \frac{1}{2} (\pi - \theta) + \frac{1}{2} \sin 2\theta R^2 \frac{dB}{dt} \]

\[ \varepsilon_{AB} = \frac{1}{2} \left( \theta - \frac{1}{2} \sin 2\theta \right) R^2 \frac{dB}{dt} \]

\[ \varepsilon_{AB} = \frac{R^2}{2} \frac{dB}{dt} \left( \sin 2\theta + \pi - 2\theta \right) \]  \hspace{1cm} (14)

The concept of probability is used in the second calculation. We might call it "the probability method". To compare equation 14 and 8, it can be seen that "probabilistic method" and the "integration method" fully consistent with the results.

Demonstrated above is the intersection of wires and the magnetic field. There are two relative positions between the "infinitely long" straight wire and the magnetic field. The first, wire and magnetic fields do not intersect, that is, the wire is posed outside the magnetic field[3]; and the second, the wire is through the center of a circular region of magnetic field. To do the similar calculation, it is not difficult to prove that for both cases, results of the "probability method" and the "integration method" are same. By now, we have proved that using "probabilistic method" is correct for solving this problem.

According to the principle of mathematical statistics, the EMF at both ends of wire AB should be expected as:

\[ \varepsilon_{AB} = p_1 \varepsilon_1 + p_2 \varepsilon_2 \]  \hspace{1cm} (13)

From the above calculation we can see that the actual direction of \( \varepsilon_1 \) is from B to point A, reversed to \( \varepsilon_{AB} \). The actual direction of the \( \varepsilon_2 \) is from A to point B, the same to \( \varepsilon_{AB} \). Therefore, in the calculation \( \varepsilon_1 < 0, \varepsilon_2 > 0 \) should be taken. The probability of the existence of the two loops are 1/2 each. Therefore, \( p_1 = p_2 = 1/2 \), are substituted into equation 13 and taking into account equation 10 and 12, the electromotive force at "unlimited long" straight wire AB both ends is

\[ \varepsilon_{AB} = \frac{1}{2} (\pi - \theta) + \frac{1}{2} \sin 2\theta R^2 \frac{dB}{dt} \]

\[ \varepsilon_{AB} = \frac{1}{2} \left( \theta - \frac{1}{2} \sin 2\theta \right) R^2 \frac{dB}{dt} \]

\[ \varepsilon_{AB} = \frac{R^2}{2} \frac{dB}{dt} \left( \sin 2\theta + \pi - 2\theta \right) \]  \hspace{1cm} (14)

The concept of probability is used in the second calculation. We might call it "the probability method". To compare equation 14 and 8, it can be seen that "probabilistic method" and the "integration method" fully consistent with the results.

Demonstrated above is the intersection of wires and the magnetic field. There are two relative positions between the "infinitely long" straight wire and the magnetic field. The first, wire and magnetic fields do not intersect, that is, the wire is posed outside the magnetic field[3]; and the second, the wire is through the center of a circular region of magnetic field. To do the similar calculation, it is not difficult to prove that for both cases, results of the "probability method" and the "integration method" are same. By now, we have proved that using "probabilistic method" is correct for solving this problem.
Conclusion

In summary, for above-mentioned problems of “unlimited long” straight wire and circular symmetric uniform magnetic field, no matter what their relative positions is, we can find quickly and easily the induction electromotive force in the wire using "probabilistic method". The first reason is the particularity of the "unlimited long" straight wire; secondly, Faraday's law of electromagnetic induction is universal applicable. When using "probabilistic method" to solve such problems, there is no need to know the specific expression of the electric field induced $\mathbf{E}_i$. Thus the complicated integral calculations is avoided. The method is with clear physical image, innovative problem-solving ideas, and simple and quick calculation. The author presented this approach in the teaching, which was welcomed by students. What is more important is that the new approach has also expanded the direct use of Faraday's law of electromagnetic induction for solving the problems of electromotive force. This would be an attempt to enhance the understanding of the concepts and creative thinking ability of the students. This attempt has also a positive meaning today when scientific and technological innovation are given much emphasis on quality education.

References

