Overall Excess Burden Minimization from a Mathematical Perspective

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Abstract. The principle of overall excess burden minimization is an important part in the field of public finance, and the authors of most public finance textbooks do not offer a rigorous mathematical proof. To prove this principle, this study develops algebraic and geometric methods to examine the links between tax revenue and excess burden in a general case and further analyzes the dynamic process of tax revenue and excess burden when the tax rate is changed. Finally, the optimal overall excess burden minimization is demonstrated to verify the principle in terms of the concavity of the excess burden curve.

Introduction

In the theory of optimal commodity taxation, overall excess burden is inevitable to some extent. Theoretically, the objective of optimal commodity taxation relies on minimizing the excess burden for a certain amount of tax revenue. A principle is established to minimize overall excess burden that is the prerequisite of the Ramsey Rule. Rosen (2005)[1] points out the principle, "marginal excess burden of the last dollar of revenue raised from each commodity must be same", and the dynamic path of evolvement of excess burden for its minimization, "otherwise, it would be possible to lower overall excess burden by raising the rate on the commodity with the smaller marginal excess burden, and vice versa." Using a mathematical method to logically verify this principle is crucial for not only deriving the Ramsey Rule but also justifying the theory of optimal commodity taxation. This study develops an algebraic and geometrical method that certifies each other to show the dynamic procedure of the abovementioned principle.

Two Methods Verifying the Principle of Minimizing Overall Excess Burden

There is the equivalence of the algebraic and geometric representations of decreasing excess burden (Trandel, 2003) [3]. Algebraic method provides a clear, logical process, and geometric ones visually illustrate relationships. Therefore, the two methods are complementary. Stiglitz (2000)[2], Trandel (2003)[3], and Rosen (2005)[1] have shown that the two methods have identical effectiveness.

Here, we assume two commodities, X and Y, have the competitive prices of $P_X$ and $P_Y$, respectively. The tax authority imposed a unit tax on X and Y of $t_X$ and $t_Y$ respectively. The excess burdens will occur for X and Y without considering the leisure factor and lump sum tax. The theory of optimal commodity tax aims to minimize the overall excess burdens although it cannot be eliminated completely. TR and EB represent tax revenue and excess
burden, respectively. The overall excess burden has two parts: \( EB_X \) for \( X \) and \( EB_Y \) for \( Y \), as in Eq. 1, and is restricted by consumer's income spent on a certain amount of \( X \) and \( Y \); that is, overall excess burden is subject to the price and quantity consumed of \( X \) and \( Y \) with a tax.

Furthermore, we suppose that all the income is spent only on the two commodities in the amount of \( Q_X \) and \( Q_Y \), respectively. Therefore, income is determined by the consumption, commodity price, and unit tax.

\[
\min(EB = EB_X + EB_Y).
\]

S.t.: 
\[ I = I(t_X, P_X, Q_X; t_Y, P_Y, Q_Y) \]

We discuss how the tax revenue affects excess burden before attempting optimization. According to the occurrence of excess burden, generally, the excess burden increases when tax revenue increases, while the rate of increase in excess burden exceeds that for tax revenue.

Hence, 
\[
\frac{d(EB)}{d(TR)} > 0 \quad \text{and} \quad \frac{d^2(EB)}{d(TR)^2} > 0.
\]

The analysis on the links of \( EB \) and \( TR \) is critical to deduce the optimal result.

**Links between \( TR \) and \( EB \)**

A general case is introduced to show \( TR \) and \( EB \) if tax is changed. Assuming the demand and supply curves are \( Q^d = a - b \cdot P \) and \( Q^s = -c + d \cdot P \), (\( a, b, c, d > 0 \)), respectively, the formula \( Q^{c2} = -(c + d \cdot t_X) + d \cdot P \) is derived for the new equilibrium if \( t_X \) is imposed on commodity \( X \). According to this case, we define different coefficients and intercepts 
\[ a = 50, b = 5, c = 10, d = 5. \]

Therefore, \( TR = 20t_X - \frac{5}{2}t_X^2 \), \( EB = \frac{5}{4}t_X^2 \), as derived by plugging the related numbers into the equations for \( TR \) and \( EB \).

The objective of formulating an expression for \( TR \) and \( EB \) is to show the link between \( TR \) and \( EB \) clearly. A table and diagram show this link. Table 1 describes change in value of \( TR \) and \( EB \) with the change in \( t_X \). \( TR \) increases when \( t_X \) increases 0 to 4, and decreases when \( t_X \) increases from 4 to 8. The relationship is symmetrical because \( TR \) is a quadratic function with a maximum value of 40. \( EB \) increases steadily with increases in \( t_X \). We define \( \mu = EB/TR \). A higher \( \mu \) implies that \( EB \) increases at a higher rate than \( TR \) with \( t_X \).

Table 1 also uses the values of \( \frac{d(EB)}{d(TR)} \) and \( \frac{d^2(EB)}{d(TR)^2} \) to reveal the same. We derive
\[
\frac{d(EB)}{d(TR)} > 0 \quad \text{and} \quad \frac{d^2(EB)}{d(TR)^2} > 0 \quad \text{if} \quad t_X \in [0, 4). \]

The latter is always greater than zero. This result shows that \( EB \) increases with \( TR \) and at a faster rate than \( TR \), consistent with intuition. We also obtain the algebraic results
\[
\frac{d(EB)}{d(TR)} = \frac{t_X}{8 - 2t_X} > 0 \quad \text{and} \quad \frac{d^2(EB)}{d(TR)^2} = \frac{1}{20 \sqrt{(4 - \frac{1}{10}TR)^3}} > 0 \quad \text{if} \quad t_X \in [0, 4). \]

These results are consistent with the geometrical ones.
### Table 1. Value of $TR$ and $EB$ with the Different $t_X$

<table>
<thead>
<tr>
<th>$t_X$</th>
<th>$TR$</th>
<th>$EB$</th>
<th>$\mu = EB/TR$</th>
<th>$\frac{d(EB)}{d(TR)}$</th>
<th>$\frac{d^2(EB)}{d(TR)^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>$\infty$</td>
<td>0</td>
<td>0.00625</td>
</tr>
<tr>
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<td>0.0714</td>
<td>0.1667</td>
<td>0.00649</td>
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<tr>
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<td>0.167</td>
<td>0.5</td>
<td>0.00675</td>
</tr>
<tr>
<td>3</td>
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<td>11.25</td>
<td>0.3</td>
<td>1.5</td>
<td>0.00703</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>20</td>
<td>0.5</td>
<td>$\infty$</td>
<td>0.00732</td>
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<tr>
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<tr>
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<td>-1.5</td>
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<tr>
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<td>3.5</td>
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</tr>
<tr>
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<td>80</td>
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<td>0.00873</td>
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</tbody>
</table>

![The link of TR and EB](image)

**Figure 1.** The link of $TR$ and $EB$.

In figure 1, the geometrical trend shows that $\frac{d(EB)}{d(TR)}>0$ and $\frac{d^2(EB)}{d(TR)^2}>0$ for EB values corresponding to $t_X$ from 0 to 4. $TR$ does not decrease in practice for $t_X$ ranging from 4 to 8. According to the figure, EB is a quadratic function of TR. The maximum value of TR is 40.
EB always has a positive value. The shape of curve is similar to the Laffer curve, which shows the existence of an optimal tax that may be 2.667 when the difference between TR and EB is maximized in this case. Certainly, the most important result is \( \frac{d(EB)}{d(TR)} > 0 \) and \( \frac{d^2(EB)}{d(TR)^2} > 0 \). This result emphasizes the importance of analyzing the principle of minimizing overall EB, as done in the next section.

**Deduction of the Principle of Minimizing Overall EB**

Figure 2 is a segment of the curve in figure 1 for the range of TR from 0 to 40 when \( t_x \in [0,4) \). EB increases faster when TR increases. The curve has a positive slope and the second derivative is positive with the concave shape. Meanwhile, the analysis of the tax impact on commodity Y is similar to that of X.

The minimization condition for overall excess burden is that marginal excess burdens for last one dollar of tax revenue are the same for X and Y (Rosen, 2005)[1]. The mathematical expression is as in Eq. 2.

\[
\frac{d(EB_x)}{d(TR_x)} = \frac{d(EB_y)}{d(TR_y)}.
\]  

(2)

First, the total TR, including TR\(_x\) and TR\(_y\), is fixed, and \( \frac{d(EB_x)}{d(TR_x)} \) is not necessarily equal to \( \frac{d(EB_y)}{d(TR_y)} \) at the beginning point c and point a in figure 2. For simplification, \( \frac{d(EB_x)}{d(TR_x)} < \frac{d(EB_y)}{d(TR_y)} \) because the slope at point c is flatter than that at point a.

Second, we can decrease overall excess burden through an increase in tax rate for X. The tax revenue for X will increase until TR\(_x\) equals to TR\(_*\) as \( t_x \) increases. The excess burden for X increases from EB\(_x\) to EB\(_*\) as shown by a dark grey arrow. The personal income \( I \) is fixed, and should be allocated and spent between X and Y before tax increases in terms of consumer theory. The income spent on X increases after commodity tax for X increases. However, the consumer wants to maintain his utility unchanged when \( t_x \) increases. Therefore, the consumption of Y will decline. This causes tax revenue for Y to decrease until TR\(_y\) equals TR\(_*\) and further causes excess burden for Y to decrease from EB\(_y\) to EB\(_*\) as shown by the light grey arrows.

Now we can maintain TR\(_*\) exactly between TR\(_x\) and TR\(_y\). This means TR\(_*\) is the average value of TR\(_x\) and TR\(_y\). We get \( 2TR^* = TR_x + TR_y \) to maintain the total tax revenue unchanged, but EB\(_y\) will decrease by more than the increase in EB\(_x\), so EB\(_*\) is located close to EB\(_x\), and far from EB\(_y\), which implies \( 2EB^* < EB_x + EB_y \). The tax authority can maintain the tax revenue and decrease the excess burden simultaneously until point a meets point c at point b along the line EB=f(TR). The slope at b is the same as that of line l. This means \( \frac{d(EB_x)}{d(TR_x)} = \frac{d(EB_y)}{d(TR_y)} \). This verifies equation (2). Hence, the principle of overall excess
burden minimization in public finance is established algebraically as well as geometrically.

![Diagram of EB minimization]

**Figure 2. Dynamic process of EB minimization.**

**Conclusion**

First, this study analyzes the relationships among tax rate, tax revenue, and excess burden. Algebraically and geometrically, we obtain an important result that $\frac{d(EB)}{d(TR)} > 0$ and $\frac{d^2(EB)}{d(TR)^2} > 0$ for a relevant range of tax.

Second, we address the ambiguity in classic textbooks regarding the principle of overall excess burden minimization for two different commodities, $X$ and $Y$, which are consumed and levied by a tax. Faced with the common questions in public finance textbooks, the strict mathematical proofs are provided for the principle of overall excess burden minimization. We get the principle of overall excess burden minimization, $\frac{d(EB_x)}{d(TR_x)} = \frac{d(EB_y)}{d(TR_y)}$ and the logical process to lower overall excess burden by raising the rate on the commodity with the smaller marginal excess burden because the curve $EB=f(TR)$ is concave.

**Reference**

