Formulation of Total Shortening Distance for Obtaining an Optimal Level of Adding Short Relations to a Delegate in a Pyramid Organization

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Abstract. We study models of adding relations between a delegate member and other member of a pyramid organization for the purpose of revealing optimal additional relations. This paper proposes a model of adding edges between a delegate node and every other node of the same depth $N$ in a complete binary tree of height $H$ when adding edges are shorter than those of the complete binary tree. The lengths of adding edges are $L$ ($0 < L < 1$) while those of edges of the complete binary tree are 1. The total shortening distance to obtain the optimal depth $N^*$ which maximizes the total shortening distance is formulated. Furthermore, the total shortening distance of this model is illustrated with numerical examples.

Introduction

The pyramid organization structure can be expressed as a rooted tree, if we let nodes and edges in the rooted tree correspond to members and relations between members in the organization respectively. Then the number of children of each node in the rooted tree is the number of subordinates of each member in the organization and the height of the rooted tree is the number of levels in the organization [1, 2].

For a model of adding edges between a delegate node and every other node of the same depth $N$ in a complete binary tree of height $H$, we have obtained an optimal depth to maximize the total shortening distance such that the communication of information between every member in the organization becomes the most efficient [3, 4]. A complete binary tree is a rooted tree in which all leaves have the same depth and all internal nodes have two children [5]. The total shortening distance is the sum of shortened lengths of shortest paths between every pair of all nodes by adding edges. The above model is expressed as all edges have the same length. However, we should consider that adding edges differ from those of complete binary tree in length.

This study proposes a model of adding edges between a delegate node and every other node of the same depth $N$ ($N = 1, 2, ..., H$) in a complete binary tree of height $H$ ($H = 1, 2, ...$) when adding edges are shorter than those of the complete binary tree. The lengths of adding edges are $L$ ($0 < L < 1$) while those of edges of the complete binary tree are 1. The total shortening distance to obtain the optimal depth $N^*$ is formulated. Furthermore, the total shortening distance of this model is illustrated with numerical examples.

Formulation of Total Shortening Distance

Let $S_H(N)$ denote the total shortening distance, when we add edges between a delegate node and every other node of the same depth $N$ ($N = 1, 2, ..., H$) in a complete binary tree of height $H$ ($H = 1, 2, ...$). The lengths of adding edges are $L$ ($0 < L < 1$) while those of edges of complete binary tree are 1.

The total shortening distance $S_H(N)$ can be formulated by adding up the following three sums of shortening distances: (i) the sum of shortening distances between every pair of nodes whose depths are equal to or more than $N$, (ii) the sum of shortening distances between every pair of nodes whose...
depths are less than \(N\) and those whose depths are equal to or more than \(N\) and (iii) the sum of shortening distances between every pair of nodes whose depths are less than \(N\) in the same way as [3, 4].

The sum of shortening distances between every pair of nodes whose depths are equal to or more than \(N\) is given by

\[
A_N(H) = \{M(H - N)\}^2 (2^N - 1)L + \{M(H - N)\}^2 2^N \sum_{i=1}^{N} (i-L)2^{i-1},
\]

(1)

where \(M(h)\) denotes the number of nodes of a complete binary tree of height \(h\) \((h = 0, 1, 2, \ldots)\). The sum of shortening distances between every pair of nodes whose depths are less than \(N\) and those whose depths are equal to or more than \(N\) is given by

\[
B_N(H) = M(H - N)\{M(N-1) - N\}L + M(H - N)\sum_{i=1}^{N-1} i2^i + 2M(H - N)2^N \sum_{i=1}^{N-1} (i-L)(N-i)2^{i-1},
\]

(2)

and the sum of shortening distances between every pair of nodes whose depths are less than \(N\) is given by

\[
C(N) = \sum_{i=0}^{N-2} (N-i-1)2^i + 2^N \sum_{i=1}^{N-2} (j-L)(i-j+1)2^{j-1},
\]

(3)

where we define \(\sum_{i=1}^{0} \cdot = 0\) and \(\sum_{i=1}^{1} \cdot = 0\).

From Eq. 1, Eq. 2 and Eq. 3, the total shortening distance \(S_H(N)\) is given by

\[
S_H(N) = A_N(H) + B_N(H) + C(N)
\]

\[
= \{M(H - N)\}^2 (2^N - 1)L + \{M(H - N)\}^2 2^N \sum_{i=1}^{N} (i-L)2^{i-1}
\]

\[
+ M(H - N)\{M(N-1) - N\}L + M(H - N)\sum_{i=1}^{N-1} i2^i + 2M(H - N)2^N \sum_{i=1}^{N-1} (i-L)(N-i)2^{i-1}
\]

\[
+ \sum_{i=0}^{N-2} (N-i-1)2^i + 2^N \sum_{i=1}^{N-2} (j-L)(i-j+1)2^{j-1}.
\]

(4)

Numerical Examples

Table 1 and Table 2 illustrate the total shortening distance \(S_H(N)\) for \(H = 1, 2, \ldots, 8\) and \(N = 1, 2, \ldots, H\). Table 1 and Table 2 show \(S_H(N)\) in the case of \(L = 0.3\) and \(L = 0.8\) which are the lengths of adding edges, respectively.

Table 1 and Table 2 reveal that \(N^* = H\) maximizes \(S_H(N)\) for \(H = 1, 2, \ldots, 8\) when \(L = 0.3\) and \(L = 0.8\). These mean that the most efficient level of forming relations between a delegate member and every other member in this model is the lowest level, irrespective of the number of levels in the organization structure.

Table 1. Total shortening distance \(S_H(N)\) in the case of \(L = 0.3\).

<table>
<thead>
<tr>
<th>(N)</th>
<th>(H = 1)</th>
<th>(H = 2)</th>
<th>(H = 3)</th>
<th>(H = 4)</th>
<th>(H = 5)</th>
<th>(H = 6)</th>
<th>(H = 7)</th>
<th>(H = 8)</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1.7</td>
<td>15.3</td>
<td>83.3</td>
<td>382.5</td>
<td>1633.7</td>
<td>6747.3</td>
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<td>2</td>
<td>-</td>
<td>23.8</td>
<td>175.2</td>
<td>893.2</td>
<td>3990.0</td>
<td>16826.8</td>
<td>69073.2</td>
<td>279857.2</td>
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<tr>
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<td>-</td>
<td>208.5</td>
<td>1340.9</td>
<td>6516.9</td>
<td>28513.7</td>
<td>119086.5</td>
<td>486548.9</td>
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<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1452.0</td>
<td>8471.8</td>
<td>39707.4</td>
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<td>708609.0</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>-</td>
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<td>-</td>
<td>8805.9</td>
<td>47815.5</td>
<td>217987.5</td>
<td>926942.7</td>
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<tr>
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<tr>
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<td>-</td>
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</table>
Table 2. Total shortening distance $S_{tl}(N)$ in the case of $L = 0.8$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$H = 1$</th>
<th>$H = 2$</th>
<th>$H = 3$</th>
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<td>16.8</td>
<td>127.2</td>
<td>655.2</td>
<td>2940.0</td>
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<tr>
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<td>-</td>
<td>-</td>
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<td>5138.4</td>
<td>22623.2</td>
<td>94764.0</td>
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</table>

Conclusion

This study considered obtaining an optimal depth $N^*$ of which edges between a delegate node and every other node in a complete binary tree of height $H$ are added. In this paper the lengths of adding edges are $L$ ($0 < L < 1$) while those of edges of the complete binary tree are 1. The total shortening distance which is the sum of shortened lengths of shortest paths between every pair of all nodes by adding edges was formulated. Furthermore, the total shortening distance of this model was illustrated with numerical examples.

References


