Estimation of the Period Value at Risk of Investments with Multiple Risk Factors

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Abstract. This study is to propose a Monte Carlo simulation based method for estimating the Period Value at Risk (PVaR) of investments with multiple risk factors. There proposed many measures for market risk like variance, value at risk (VaR), however, these measures only reflect the risk at specific point of time. We proposed in 2012 the notion of period value at risk (PVaR) to reflect the risk in a time period, and suggested to estimate the PVaR of an investment using MC simulation. This paper will extend the method to estimate the PVaR of investments where multiple risk factors are involved. After proposing the MC simulation based method, we use this method to estimate the risk of an investment in five stocks assuming that the stock prices are geometric Brownian motion processes. To compare the difference with risk measured by VaR, we calculate one-year PVaR and VaR at the end of one year of the investment. Our computing experiments show that the proposed method can be used to estimate the PVaR of an investment, and there is a significant difference between PVaR and VaR. In our computing experiments, PVaR at confidence level 90\% is 0.52560 but VaR at the same confidence level is 0.17974. Hence we conclude that the proposed method is usable in estimating the PVaR of investments with multiple risk factors, and PVaR is a proper measure for market risk in a period of time.

1. Introduction

The measurement of risk is the basis of risk management. Markowitz [1] was the first to propose a measure for market risk; he suggested using variance as risk measure. Variance has been popular since the 1950s and it is the cornerstone of modern investment theories.

Variance is not a satisfying risk measure for two reasons: it penalizes gains and losses in the same way, and it is not connected with loss directly. Many other risk measures have been proposed since then, such as semivariance, absolute variance [2], Value at Risk [3], conditional Value at Risk [4], and maximum loss [5], refer to Dowd [6] for a survey of market risk measurements. While most of these risk measures were developed to measure downside risk, some were developed to improve the computational efficiency in solving investment models.

However, these risk measures only show market risk at a specified future time and fail to account for the risk in a period of time. When estimating market risk of an investment by conventional risk measures, risk factors are regarded as stochastic variables, e.g., considering stock price at a specified time as a stochastic variable, market risk is calculated by the statistical features of the investment profit. Risk estimated this way only reflects the risk at that specified time and fails to reflect the risk in a period of future time.

In real investment situations, investors may not have a clear idea on the exact period of their investments, they may tell rough period over which they intend to hold the financial instruments, and they may sell the instruments at any time during that period, only just at the end of that period. Investors in such situations care about the market risk throughout this time period, not just the risk at the end of the period, but conventional risk measures fail to account for the risk during a time period.
Xu and Huo [7] proposed the notion of Period Value at Risk (PVaR) as a measure of the risk over a period of time in the future; Huo, Xu and etc. [8] suggested a simulation based method for estimating PVaR of an investment when only one risk factor is involved.

This study will extend the method to the cases where multiple risk factors are involved in investments; this method will provide a practical method to estimate the PVaR of a portfolio with multiple financial instruments.

2. PVaR and its estimation

2.1 Definition of PVaR

PVaR of investment $x$ is defined as the highest possible loss rate that may occur with a certain probability within a time period, which is expressed as follows,

$$ PVaR_{1-\alpha}(x) = \inf\{d \in \mathbb{R}^1 \mid P(L_t(x) > d) \leq \alpha \} $$

where $L_t(x) = \max\{L(x, \omega(t)) : t \in [0,T]\}$. $L(x, \omega(t))$ is the loss rate of investment $x$ which is determined by risk factor $\omega(t)$, and $L_t(x)$ is the maximum loss rate of investment $x$ over a period $[0,T]$. $PVaR_{1-\alpha}(x)$ is the highest possible loss rate that may occur at probability $\alpha$, it is considered as an indicator of the risk over period $[0,T]$. Refer to Xu and Huo [7] for details about PVaR.

Equation (1) only defines an investment’s PVaR, its calculation is challenging. Because it is hard to give an analytical expression of PVaR when the risk factor is a general stochastic process, Huo etc. [8] proposed a numerical method to estimate PVaR by Monte Carlo simulation for the case where there is only one risk factor in the investment. This paper will extend this approach to estimate the PVaR of an investment with multiple risk factors, a portfolio with more than two financial instruments included is such an instance.

2.2 Estimation of PVaR with Monte Carlo Simulation

Consider a portfolio consisted of several correlated stocks; the risk factors are the stock prices in the future. To estimate the PVaR of such a portfolio, we make an assumption about the values of the uncertain factors as follows:

**Assumption 1** Future prices of the stocks during a time period can be described as a multi-dimensional geometric Brownian motion.

That is, the price of stock $i$, denoted by $S_i(t)$, $i = 1, 2, \ldots, n$, is the solution of the following differential equation,

$$ dS_i(t) = \mu S_i(t) dt + \sigma_i dW_i(t), \quad i = 1, 2, \ldots, n $$

where $\mu_i$ and $\sigma_i$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$) are parameters, and $(W_1(t), W_2(t), \ldots, W_n(t))$ is a $n$-dimensional standard Brownian motion.

Under Assumption 1, let the purchase prices of stock $i$ be $S_i(0), \ i = 1, 2, \ldots, n$, then prices of stock $i$ can be expressed by the following formula (Nishida, 2005)[9]:

$$ S_i(t) = S_i(0) \exp[\{\mu_i - \frac{1}{2} \sigma_i^2\} t + \sum_{j=1}^{n} \sigma_{ij} B_j(t)], \quad i = 1, 2, \ldots, n $$

Hence we can simulate the prices of stock $i$ during the time period using (3). After getting data of risk factors during the time period, we can calculate the loss rates of the portfolio in each scenario at different time spots in the period, and the largest loss rate in each scenario. Similar with the method for single risk factor case proposed in [8], we can estimate the PVaR of the portfolio as follows:

**Step 1.** Simulate the prices of stock $i$, $S_i(t), \ t \in [0,T]; i = 1, 2, \ldots, n$;

**Step 2.** Compute the loss of a portfolio using the prices of the stocks, let it be $L(t), \ t \in [0,T]$;

**Step 3.** Compute the largest loss of $L(t), \ t \in [0,T]$, let it be $\max\{L(t), t \in [0,T] \}$;
Step 4. Repeat step 1 to step 3 N times, let \( L^k_T \) be the data of \( L_T \) in the \( k \)th time, we get
\[ N \text{ data } \{ L^k_T ; k = 1, 2, \cdots, N \}; \]

Step 5. The \([1 - \alpha]N\)th smallest value in set \( \{ L^k_T ; k = 1, 2, \cdots, N \} \) is \( PVaR_{1-\alpha} \).

So we can estimate PVaR if we can simulate the risk factors in Step 1 because computations in Step 2 - Step 5 are straightforward.

Under Assumption 1, with the initial condition \( S_i(t)\big|_{t=0} = S_i(0), i = 1, 2, \cdots, n \), the solution of (2) is given by,
\[
S_i(t) = S_i(0) \exp(\left[\mu_i - 0.5(\sigma^2_{i1} + \cdots + \sigma^2_{ij} + \cdots \sigma^2_{in})\right]t + \sigma_{i1}B_i(t) + \cdots + \sigma_{ij}B_j(t) + \cdots + \sigma_{in}B_n(t) ) \]
\[
\ldots
\]
\[
S_n(t) = S_n(0) \exp(\left[\mu_n - 0.5(\sigma^2_{n1} + \cdots + \sigma^2_{nj} + \cdots \sigma^2_{nn})\right]t + \sigma_{n1}B_1(t) + \cdots + \sigma_{nj}B_j(t) + \cdots + \sigma_{nn}B_n(t) ) \]
\[
(4)
\]

In order to simulate \( S_i(t), i = 1, 2, \cdots, n \) using (4), it is necessary to determine the parameters in (4). We next suggest a method to determine these parameters using historical price data of these stocks.

Suppose we have \( D+1 \) sequential close daily prices of each stock in past \( T \) years, denoted by \( S_i(t_1), S_i(t_1^1), S_i(t_1^2), \cdots, S_i(t_N) \), \( i = 1, 2, \cdots, n \). The elapsed time between any two sequential time spots is \( \Delta t = T/1 \). The rate of profit in two adjacent times, measured by the logarithm yield, is
\[
r_i(t_d) = \ln \left( S_i(t_1^d) / S_i(t_{d-1}) \right), d = 1, 2, \cdots, D; i = 1, 2, \cdots, n .
\]

(5)

Define the following notations:

- Ei: expectation of \( r_i(t_d^1), d = 1, 2, \cdots, D; i = 1, 2, \cdots, n \);
- Vi: variance of \( r_i(t_d^1), d = 1, 2, \cdots, D; i = 1, 2, \cdots, n \);
- \( C_{ij} \): covariance of \( r_i(t_d^1) \) and \( r_j(t_d^1), d = 1, 2, \cdots, D; i, j = 1, 2, \cdots, n, i \neq j \).

Denote two adjacent points of time in \([0, T]\) with time interval \( \Delta t \) by \( t_{d-1} \) and \( t_d \), and the profit rate in the two adjacent times measured by the logarithm yield by \( r_i(t_d), i = 1, 2, \cdots, n \). Because the stock prices are supposed to be an n-dimensional geometric Brownian motion as described by (4), we have the following relationship:
\[
r_i(t_d) = \left( \left( \mu_i - 0.5(\sigma^2_{i1} + \cdots + \sigma^2_{ij} + \cdots \sigma^2_{in}) \right)\Delta t + \sigma_{i1}(B_1(t_d) - B_1(t_{d-1})) + \cdots + \sigma_{ij}(B_j(t_d) - B_j(t_{d-1})) + \cdots + \sigma_{in}(B_n(t_d) - B_n(t_{d-1})) \right)
\]
\[
\ldots
\]
\[
r_n(t_d) = \left( \left( \mu_n - 0.5(\sigma^2_{n1} + \cdots + \sigma^2_{nj} + \cdots \sigma^2_{nn}) \right)\Delta t + \sigma_{n1}(B_1(t_d) - B_1(t_{d-1})) + \cdots + \sigma_{nj}(B_j(t_d) - B_j(t_{d-1})) + \cdots + \sigma_{nn}(B_n(t_d) - B_n(t_{d-1})) \right)
\]
\[
(6)
\]

Because \( (B_1(t), B_2(t), \cdots, B_n(t)) \) is an n-dimensional Brownian motion, according to the properties of standard Brownian motion, we know that \( B_j(t_d) - B_j(t_{d-1}) \sim N(0, \Delta t); j = 1, 2, \cdots, n \). Hence, the expectation, variance and covariance of the profit rates is \[ (\mu_i - 0.5 \sum_{j=1}^{n} \sigma^2_{ij})\Delta t, \Delta t \sum_{j=1}^{n} \sigma^2_{ij}, \text{ and } \Delta t \sum_{j=1}^{n} \sigma_{ij} \sigma_{ij} \text{ respectively.} \]

On the other hand, using historical price of those stocks, we can calculate the expectation, variance and covariance of the profit rates. One way to determine \( \mu_i \) and \( \sigma_{ij}, i, j = 1, 2, \cdots, n \) is to make the values of expectation, variance and covariance calculated from (6) consistent with that estimated using the historical price data.

That is, we determine these parameters from solving the following equations:
\[ (\mu_i - 0.5\sum_{j=1}^{n}\sigma_{ij}^2)\Delta t = E_i, \quad i = 1, 2, \cdots, n \]  

(7)

\[ \Delta t \sum_{j=1}^{n}\sigma_{ij}^2 = V_i, \quad i = 1, 2, \cdots, n \]  

(8)

\[ \Delta t \sum_{i,j=1}^{n}\sigma_{ij}\sigma_{i,j} = C_{i,j}, \quad i, j = 1, 2, \cdots, n, \quad i \neq j \]

It is easy to get \( \mu_i ; i, j = 1, 2, \cdots, n \), from (7) and (8)\

\[ \mu_i = (E_i + 0.5V_i)/\Delta t, \quad i = 1, 2, \cdots, n \]

Equations (8) have \( n^2 \) variables and \( (n^2 + n)/2 \) equations; its solution is not unique. One way to solve these equations is using the Cholesky decomposition.

Let

\[
A = \frac{1}{\Delta t} \begin{bmatrix} V_1 & C_{12} & \cdots & C_{1n} \\ C_{12} & V_2 & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & V_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}
\]

Since, equations (8) can be written as

\[ BB' = A \]

(9)

If matrix \( A \) has real entries and is symmetric (or more generally, has complex-valued entries and is Hermitian) and positive definite, then matrix \( A \) can be decomposed as the product of two matrixes \( BB' \), where \( B \) is a lower triangular matrix with strictly positive diagonal entries, and \( B' \) denotes the conjugate transpose of \( B \). This is the Cholesky decomposition; refer to Wilkinson [10] for details.

If the matrix \( A \) is symmetric and positive definite, we can get a solution of \( B \) by doing Cholesky decomposition.

If \( A \) can not be decomposed, we determine the parameters \( \sigma_{ij}, i, j = 1, 2, \cdots, n \) by minimizing the difference between the theoretical value and historical value of the variance and covariance, that is, we determine the parameters by solving the following minimization model,

\[
\min \sum_{i=1}^{n}(V_i - V(r_i(t_d)))^2 + \sum_{i,j=1}^{n}(C_{i,j} - C(r_i(t_d), r_j(t_d)))^2
\]

(10)

where \( V(r_i(t_d)) \) and \( C(r_i(t_d), r_j(t_d)) \) is the theoretical value of the variance and covariance, and \( V_i \) and \( C_{i,j} \) is the historical value.

3. Experiments of risk estimation

In order to illustrate the method proposed in previous section, we estimate the risk of an investment in five components of the Dow Jones Industrial Average; they are AXP, BA, CAT, DD and DIS. Suppose the portfolio is composed by the five stocks with one share from each stock.

We use the daily close prices of the five stocks in 2015/9/1-2016/8/31 in estimating parameters needed in simulation for future prices, which is downloaded from Yahoo! Finance, and suppose that the investment term begins on 2017/6/1 and will not be longer than one year.

Since the portfolio is composed of five stocks, we use five independent Brownian motions to express the future prices of the five stocks. We use the Cholesky decomposition to decompose the correlation matrix for determining the parameters, the results are shown in Table 1.
Table 1. Parameters estimated from historical price data.

<table>
<thead>
<tr>
<th>( \hat{\mu}_i )</th>
<th>( \hat{\sigma}_{ij} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.07373</td>
<td>0.23943</td>
</tr>
<tr>
<td>0.07970</td>
<td>0.09933</td>
</tr>
<tr>
<td>0.17348</td>
<td>0.10580</td>
</tr>
<tr>
<td>0.39341</td>
<td>0.08323</td>
</tr>
<tr>
<td>-0.01898</td>
<td>0.05476</td>
</tr>
</tbody>
</table>

We create 100 thousands scenarios for each price process of the five component stocks, and estimate the PVaR with the method described in previous. We calculate the PVaR of the portfolio at confidence level 85%, 90% and 95%, the results are shown in Table 2.

Table 2. PVaR and VaR estimated by the proposed method.

<table>
<thead>
<tr>
<th>Confidence level</th>
<th>85%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>PvaR</td>
<td>0.43978</td>
<td>0.52560</td>
<td>0.65544</td>
</tr>
<tr>
<td>VaR</td>
<td>0.03454</td>
<td>0.17974</td>
<td>0.38039</td>
</tr>
</tbody>
</table>

To compare market risk measured by different measures, we calculate the VaR at the same confidence level of this portfolio at the end of one year beginning from 2017/6/1. We use the last price of each simulated scenario as the price sample of the corresponding stock at end of this period, thus we get 100 thousands price samples for each component stock of the portfolio. Then we calculate the loss of the portfolio using these price data, producing 100 thousands samples of the portfolio loss, then VaR is estimated by the scenario simulation method, the corresponding VaR is also shown in Table 2.

We can see from the above experiments that risk during a period of time is much bigger that the risk at the end of that period. VaR does not reflect the risk within a time span, PVaR is a proper alternative when risk within a period of time is of concern.

4. Conclusions

This paper proposed a simulation-based method to estimate the PVaR of investments with multiple risk factors; the proposed method provides an effective method for estimating market risk in a period of future time, thus making PVaR usable in investment practices.

To illustrate the proposed method, we conducted PVaR estimation experiments taking an investment in five stocks as an example, our computing experiments indicated that the proposed method is usable in estimating the market risk of the investment during a time period.

We also compared market risk of the investment over a period of time with risk at a specified time. Our computing results indicated that PVaR in one-year is much greater than the VaR at the end of one year, indicating that PVaR is a proper indicator of risk in a time period.

This paper addressed the issue of computation of PVaR when several risk factors are involved in investments. We are working on methods for portfolio selection with PVaR as the risk indicator.

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6. References


