A Regularity Criterion for the Navier-Stokes Equations with Damping in Terms of One Directional Derivative of the Velocity Field in BMO Space

Jian-lin ZHANG$^{1,2,*}$ and Yu-ming QIN$^3$

$^1$College of Information Science and Technology, Donghua University, Shanghai 201620, P. R. China
$^2$Department of Applied Mathematics, College of Science, Zhongyuan University of Technology, Zhengzhou 450007, P. R. China
$^3$Department of Applied Mathematics, Donghua University, Shanghai 201620, P. R. China

*Corresponding author

Keywords: Navier-stokes equations, Regularity criterion, Weak solutions, Strong solutions, BMO space.

Abstract. In this paper, we consider the regularity problem for the weak solutions to the Navier-Stokes equations with damping term in $\mathbb{R}^3$. We establish a regularity criterion in terms of the derivative of the velocity in one direction for this problem in BMO space.

Introduction

We consider the following Cauchy problem for the incompressible Navier-Stokes equations with damping term $|u|^{\alpha} u$ ($\alpha \geq 0$) for $(t, x) \in (0, T) \times \mathbb{R}^3$,

$$\partial_t u - \nu \Delta u + (u \cdot \nabla)u + a |u|^{\alpha} u = -\nabla P,$$

$$\nabla \cdot u = 0,$$

$$u(0, x) = u_0(x),$$

where $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $P \in \mathbb{R}$ denote the velocity field and the pressure, respectively. $\nu$ is called the viscosity coefficient and $a$ is a nonnegative real number. This model is the so-called Brinkman-Forchheimer-exended Darcy model which comes from a porous media flow, drag or friction effects and some dissipative mechanisms (see [2, 6, 13] and references therein). Notice that in the limit case $a = 0$, we obtain the classical Navier-Stokes system.

As far as I know, Cai and Jiu [2] established the existence of weak solutions for $\alpha \geq 0$ and existence (for $\frac{5}{4} \leq \alpha < 2$) and uniqueness (for $\frac{5}{4} < \alpha \leq 2$) of strong solutions to problem (1)-(3). Markowich, Titi and Trabelsi [13] extended results in [2] and obtained existence and uniqueness of weak and strong solutions for a larger range of $\alpha$, with initial data in $H^1$. Kalantarov and Zelik [10] showed the uniqueness of solutions for $\alpha \geq 1$ with the Dirichlet boundary conditions and regular enough initial data in $H^1$. Zhou [25] proved the existence and uniqueness of global strong solutions for $\alpha \geq 1$ and gave two regularity criteria as $0 \leq \alpha < 1$. So we may assume $0 \leq \alpha < 1$.

For the classical Navier-Stokes equations, Leray [12] and Hopf [5] constructed a weak solution (so-called Leray-Hopf weak solution) for arbitrary initial data in $L^2(\mathbb{R}^3)$. Later on, huge contributions have been devoted in an effort to understand regularities of the weak solution. Different regularity criteria of the weak solutions have been proposed, such as the Prodi-Serrin conditions (see [14, 15, 19]), the Serrin's type regularity criteria on the gradient of the velocity field (see [1, 21, 23]), and so on. There are also some regularity criteria in terms of pressure, the pressure gradient and vorticity. We refer the readers to literature [7-9, 16, 17, 20, 22, 24] and references therein.
In this paper, our main purpose is to establish a new regularity criterion for the weak solutions to problem (1)-(3) in BMO space. Meanwhile, we shall show the uniqueness of the strong solutions. It is worth pointing out that our result holds for \( 0 < \alpha < 1 \) and improves the previous work in [2,30]. In addition, we also obtain the uniqueness of global strong solutions.

In what follows, we denote by \( C \) a universal positive constant, whose value may depend on \( T \) and the initial data \( u_0 \), and may change from line to line. In addition, \( L^p, 1 \leq p \leq +\infty \), and \( H^1 = W^{1,2} \) denote the usual Lebesgue spaces and Sobolev spaces on \( \mathbb{R}^3 \); \( \| \cdot \|_B \) denotes the norm in the space \( B \), \( \| \cdot \|_{C^0([0,T])} \) or \( \| \cdot \|_{L^2} \) and \( \partial_t \) and \( \partial_x \) denote the derivative with respect to \( t \) and \( x \) in the distribution sense, respectively.

We organize our present paper as follows. In Section II, we will state some crucial preliminaries and our main theorem. Subsequently, in Sections III, some crucial a priori estimates are stated and the main theorem will be proved.

**Preliminaries and Main Results**

We first recall the definition of BMO space and some crucial inequalities in the whole space \( \mathbb{R}^3 \) (see [3,4,11]).

Let \( Q \) stand for a cube with sides parallel to the axes. Denote \( f^\#_Q \) the mean oscillation of \( f \in L^1_{loc}(\mathbb{R}^3) \) in a cube \( Q \), that is,

\[
f^\#_Q = \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx
\]

where

\[
f_Q = \frac{1}{|Q|} \int_Q f(x) dx.
\]

**Definition 2.1.** (See [18]) The space \( BMO(\mathbb{R}^3) \) is defined as a function of bounded mean oscillation, i.e., \( f \in BMO(\mathbb{R}^3) \) such that

\[
\|f\|_{BMO} = \|f^\#_Q\|_{L^\infty} < +\infty
\]

where \( f^\#_Q(x) = \sup_{d>0} f^\#_{Q(x,d)} \) is the sharp function and \( Q(x,d) \) is a cube centered at \( x \) and of diameter \( l(Q) = d \).

**Lemma 2.1.** (i) The following interpolation inequality holds

\[
\|f\|_{L^q}^q \leq C\|f\|\|f\|_{BMO}.
\]  \hspace{1cm} (4)

(ii) For \( 2 \leq q \leq 6 \) and \( f \in H^1(\mathbb{R}^3) \), we have

\[
\|f\|_{L^q}^{2q} \leq C\|f\|^{6-q} \|\partial_x f\|^{q-2} \|\partial_y f\|^{q-2} \|\partial_z f\|^{q-2} \\
\leq C\|f\|^{6-q} \|f\|_{H^1}^{q-2}.
\]  \hspace{1cm} (5)

(iii) For \( 1 \leq q < +\infty \) and \( f \in H^1(\mathbb{R}^3) \), we have

\[
\|f\|_{L^q}^q \leq C\|\partial_x f\| \|\partial_y f\| \|\partial_z f\|_{L^q}.
\]  \hspace{1cm} (6)

By a weak solution, we mean \((u,P)\) satisfies (1)-(3) in the distribution sense. In addition, the following basic regularity for the weak solution
\[ u \in L^\alpha(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \]

\[ \cap L^{2\alpha_2}(0, T; L^{2\alpha_2}(\mathbb{R}^3)), \quad (7) \]

for any \( T > 0 \). If a weak solution \( u \) satisfies
\[ u \in L^\alpha(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)), \]
then \( u \) is actually a strong (classical) solution.

We are in a position to state our main results.

**Theorem 2.1.** Let \( 0 \leq \alpha < 1 \). Suppose \( u_0 \in H^1(\mathbb{R}^3) \) and \( \nabla u_0 = 0 \) in the sense of distributions. Assume that \( u(x,t) \) is a weak solution of (1)-(3) on \((0,T)\). If \( u \) satisfies

\[ \partial_s u \in L^2(0,T; BMO(\mathbb{R}^3)), \quad (8) \]

Then \( u \) is a strong solution of (1.1)-(1.3) on the interval \([0,T]\) in the sense that \( u \in C^\infty((0,T) \times \mathbb{R}^3) \). Moreover, the strong solution \( u \) is global and unique.

**Remark 2.1.** Since \( L^\infty(\mathbb{R}^3) \subset BMO(\mathbb{R}^3) \), the BMO space is the larger function class.

**Remark 2.2.** If \( a = 0 \) in problem (1)-(3), then problem (1.1)-(1.3) becomes the classical Navier-Stokes equations. Thus our result holds for the classical Navier-Stokes equations.

**The proof of Theorem 2.1**

The proof of Theorem 2.1 can be divided into three major parts. The first part establishes the bounds of \( \|u\|_2 \) and the time integral of \( \|\nabla \partial_s u\|_2 \) while the second one controls \( \|\nabla u\|_2 \) by the time integral of \( \|\nabla \partial_s u\|_2 \). Finally, we need to show the uniqueness of global strong solutions.

We start by looking for an \( L^2 \) uniform estimate for the velocity. For this purpose, we multiply (1) by \( u \) and integrate over \( \mathbb{R}^3 \) to obtain

\[ \frac{d}{dt} \|u(t)\|^2 + 2\nu \|\nabla u(t)\|^2 + 2a \|u(t)\|^{2\alpha_2}_2 = 0, \quad (9) \]

thanks to the fact that \( \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot u \ dx = 0 \). Now integrating (9) with respect to \( t \) yields

\[ \|u(t)\|^2 + 2\nu \int_0^t \|\nabla u(s)\|^2 \ ds + 2a \int_0^t \|u(s)\|^{2\alpha_2}_2 \ ds = \|u_0\|^2 \quad (10) \]

Therefore, we know that if \( u_0 \in L^2(\mathbb{R}^3) \), then for all \( t \in [0,T] \), \( u \) is a weak solution to problem (1)-(3).

**Lemma 3.1.** Suppose that \( u_0 \in H^1(\mathbb{R}^3) \) and \( \nabla u_0 = 0 \) in the sense of distributions. Let \( u(x,t) \) be a weak solution of (1)-(3) on \((0,T)\) which satisfies the energy equality (10). Assume that the condition (8) holds, then for any \( t \leq T \), we have

\[ \|\partial_s u(t)\|^2 + \int_0^t (\|\nabla \partial_s u\|^2 + \|u\|^{\alpha_2} \|\partial_s u\|^{\alpha_2}) \ ds \leq C \quad (11) \]

for some constant \( C > 0 \).

**Proof.** Multiplying (1) by \(-\partial_s^2 u\) in \( L^2(\mathbb{R}^3) \), we have

\[ \frac{1}{2} \frac{d}{dt} \|\partial_s u(t)\|^2 + \nu \|\nabla \partial_s u(t)\|^2 + a \int_{\mathbb{R}^3} (1u^{\alpha_2} u \cdot \partial_s^2 u)(x,t) \ dx \]

\[ = \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \partial_s^2 u)(x,t) \ dx. \quad (12) \]
Integration by parts gives
\[
-\int_{\mathbb{R}^3} (1|u|^{2\alpha} u \cdot \partial_s u)(x,t) \, dx = \int_{\mathbb{R}^3} (1|u|^{2\alpha} 1 \partial_s u \cdot \partial_s u)(x,t) \, dx \\
+ 2\alpha \int_{\mathbb{R}^3} [(u \cdot \partial_s)^2 |u|^{2\alpha - 2}](x,t) \, dx \\
= (1 + 2\alpha)|(|u|^{2\alpha} \partial_s u)(t)|^2.
\] 

Inserting (13) into (12) leads to
\[
\frac{d}{dt} \| \partial_s u(t) \|^2 + 2\| \nabla \partial_s u(t) \|^2 + 2a(1 + 2\alpha)(|(|u|^{2\alpha} \partial_s u)(t)|^2
\]
\[
= 2\int_{\mathbb{R}^3} [(u \cdot \nabla)u \cdot \partial_s u](x,t) \, dx := I.
\] 

Next, we estimate $I$ on the right-hand side of (14).

Applying an integration by parts and the Young inequality, we can derive from (4) that
\[
I = -2\int_{\mathbb{R}^3} [(\partial_s u \cdot \nabla u \cdot \partial_s u](x,t) \, dx \leq C\| \partial_s u(t) \|_{L^2_t}^2 \| \nabla u(t) \|
\]
\[
\leq C\| \partial_s u(t) \| \| \partial_s u(t) \|_{BMO} \| \nabla u(t) \|
\]
\[
\leq C(\| \partial_s u(t) \|^2 + 1)(\| \partial_s u(t) \|_{BMO}^2 + \| \nabla u(t) \|^2)
\] 

Inserting (15) into (14) to get
\[
\frac{d}{dt}(1 + \| \partial_s u(t) \|^2) + 2\| \nabla \partial_s u(t) \|^2 + 2a(1 + 2\alpha)(|(|u|^{2\alpha} \partial_s u)(t)|^2
\]
\[
\leq C(\| \partial_s u(t) \|^2 + 1)(\| \partial_s u(t) \|_{BMO}^2 + \| \nabla u(t) \|^2)
\] 

Thanks to the Gronwall inequality, we derive from (8), (10) and (16) that
\[
1 + \| \partial_s u(t) \|^2
\]
\[
\leq (1 + \| \partial_s u_0 \|^2) \exp \left( C\int_0^t (\| \partial_s u \|_{BMO}^2 + \| \nabla u \|^2)(s) \, ds \right)
\]
\[
\leq (1 + \| \partial_s u_0 \|^2) e^{C \| \partial_s u \|^2_{BMO} \int_0^t (s) \, ds}
\]
\[
\leq (1 + \| \partial_s u_0 \|^2) e^{C \| \partial_s u \|^2_{BMO} \int_0^t (s) \, ds}
\]
\[
\text{and}
\]
\[
\int_0^t (\| \nabla \partial_s u \|^2 + \| u \|^2 \| \partial_s u \|^2)(s) \, ds \leq C
\] 

where a constant $C$ depends on the initial data $u_0$, $\alpha$, $\nu$ and $\| \partial_s u \|_{L^2(\partial_T BMO(\mathbb{R}^3))}$. Therefore, we complete the proof of this lemma.

Now we turn to construct bounds for $H^1$ estimates.

**Lemma 3.2.** Under the same assumptions as Lemma 3.1, for any $T \leq T$, we also have
\[
\| \nabla u(t) \|^2 + \int_0^t (\| \Delta u \|^2 + \| u \|^2 \| \nabla u \|^2)(s) \, ds \leq C
\] 

where constant $C > 0$ does not only depend on $T$, $\nu$, $\alpha$ and $u_0$, but also on $\| \partial_s u \|_{L^2(\partial_T BMO(\mathbb{R}^3))}$. 

**Proof.** Multiplying (1) by $-\Delta u$ in $L^2(\mathbb{R}^3)$ and employing an integration by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u(t) \|^2 + \nu \| \Delta u(t) \|^2 + a(1 + 2\alpha)\| | u \|^2 \nabla u(t) \|^2 \\
= \int_{\mathbb{R}^3} [(u \cdot \nabla)u \cdot \Delta u](x,t)dx.
\] (20)

By the Hölder inequality, the Young inequality and (6) for \( q = 2 \), we see

\[
\int_{\mathbb{R}^3} [(u \cdot \nabla)u \cdot \Delta u](x,t)dx \leq \| \nabla u(t) \|_L^3 \leq C \| \nabla u(t) \|_L^3 \| \nabla u(t) \|_L^3 \\
\leq C \| \nabla u(t) \|_L^3 \| \nabla \partial_{x_i} u(t) \|^2 \| \nabla \partial_{x_i} u(t) \|^2 \\
\leq C \| \nabla u(t) \|_L^3 \| \nabla^2 u(t) \| \| \nabla \partial_{x_i} u(t) \|^2 \\
\leq \frac{V}{2} \| \Delta u(t) \|^2 + C \| \nabla u(t) \|_L^3 \| \nabla \partial_{x_i} u(t) \|
\leq \frac{V}{2} \| \Delta u(t) \|^2 + C \| \nabla u(t) \|_L^3 (\| \nabla u(t) \|^2 + \| \nabla \partial_{x_i} u(t) \|^2)
\] (21)

Inserting (21) into (20) to get

\[
\frac{d}{dt} \| \nabla u(t) \|^2 + \nu \| \Delta u(t) \|^2 + 2a(1 + 2\alpha)\| | u \|^2 \nabla u(t) \|^2 \\
\leq C \| \nabla u(t) \|^2 (\| \nabla u(t) \|^2 + \| \nabla \partial_{x_i} u(t) \|^2).
\] (22)

Thanks to the Gronwall inequality, (10) and Lemma 3.1, we conclude that

\[
\| \nabla u(t) \|^2 + \int_0^t (\| \Delta u \|^2 + \| u \|^2 \nabla u \|^2)(s)ds \\
\leq \| \nabla u_0 \|^2 \exp \left( C \int_0^t (\| \nabla u \|^2 + \| \nabla \partial_{x_i} u \|^2)(s)ds \right) \leq C.
\] (23)

Thus we complete the proof.

Now we shall show the continuous dependence of the solutions on the initial data, especially the uniqueness of solutions.

**Lemma 3.3.** Assume that \( u \) and \( v \) are two solutions to problem (1)-(3) in \( C(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)) \) with the same initial datum \( u_0 \in H^1(\mathbb{R}^3) \) and \( \nabla u_0 = 0 \). Then \( u \equiv v \) on \([0,T] \).

**Proof.** Let \( w = u - v \), then \( w \) satisfies

\[
\partial_t w - \nu \Delta w + (w \cdot \nabla)u + (v \cdot \nabla)w + a(| u |^{2\alpha} u - | v |^{2\alpha} v) \\
= -\nabla (P_a - P_v)
\] (24)

with \( \nabla \cdot w = 0 \).

Now multiplying (24) by \( w \), we easily obtain

\[
\frac{1}{2} \frac{d}{dt} \| w(t) \|^2 + \nu \| \nabla w(t) \|^2 + a \int_{\mathbb{R}^3} [(| u |^{2\alpha} u - | v |^{2\alpha} v) \cdot w](x,t)dx \\
= -\int_{\mathbb{R}^3} [(w \cdot \nabla)u \cdot w](x,t)dx
\] (25)

where we have used \( \int_{\mathbb{R}^3} (v \cdot \nabla)w \cdot wdx = 0 \).
By virtue of the Hölder inequality, the Young inequality and (5) for $q = 6$ and for $q = 3$, we have

$$\leq C \| w(t) \| \| \nabla w(t) \| \| \nabla u(t) \|^{\frac{1}{6}}$$

$$\times \| \nabla \partial_{x_j} u(t) \|^{\frac{1}{6}} \| \nabla \partial_{x_j} u(t) \|^{\frac{1}{6}} \| \nabla \partial_{x_j} u(t) \|^{\frac{1}{6}}$$

$$\leq C \| w(t) \| \| \nabla w(t) \| \| \nabla u(t) \|^{\frac{1}{2}} \| \Delta u(t) \|^{\frac{1}{2}} \| \nabla \partial_{x_j} u(t) \|^{\frac{1}{6}}$$

$$\leq \frac{V}{2} \| \nabla w(t) \|^2 + C \| w(t) \|^2$$

$$\times (\| \nabla u(t) \|^2 + \| \Delta u(t) \|^2 + \| \nabla \partial_{x_j} u(t) \|^2).$$

In addition, it is well known that there is a nonnegative constant $k = k(\alpha)$ such that

$$(u - v) \cdot ((u - v) \cdot v) \geq k(\alpha) |u - v|^2 (|u| + |v|)^{2\alpha} \geq 0,$$

which implies

$$\int_{\mathbb{R}^3} \left( (|u| + |v|)^{2\alpha} |w|(x,t) dxight. \left. \geq k \int_{\mathbb{R}^3} \left( |w|^2 (|u| + |v|)^{2\alpha} \right) (x,t) dx \right.$$

$$= k \| (|u| + |v|)^{2\alpha} |w|(t) \|^2.$$  

Inserting (3.18)-(3.19) into (3.17), we have

$$\frac{d}{dt} \| w(t) \|^2 + \nu \| \nabla w(t) \|^2 + 2ak \| (|u| + |v|)^{\alpha} |w|(t) \|^2$$

$$\leq C (\| \nabla u(t) \|^2 + \| \Delta u(t) \|^2 + \| \nabla \partial_{x_j} u(t) \|^2) \| w(t) \|^2.$$  

Applying the Gronwall inequality, (10), Lemmas 3.1 and 3.2 and thanks to $w(0) = 0$, we obtain that $u = v$ on $[0, T]$. Thus we complete the proof.

Finally, we are in a position to state the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Similarly to [2,25], and by Lemmas 3.1-3.3, we know that there is a uniqueness strong solution $u$ in the class of weak solutions to problem (1)-(3) satisfying for some

$$T^* > 0$$

and any given $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $u \in C([0,T^*), H^1(\mathbb{R}^3)) \cap C^1(0,T^*), H^1(\mathbb{R}^3)) \cap C((0,T^*), H^1(\mathbb{R}^3))$. In addition, by the condition (8) and standard continuation argument, the local solution can be extended to time $T$. So we have shown that $u$ actually is a strong solution on $[0,T]$. This completes the proof of Theorem 2.1.

**Acknowledgements**

This work was in part supported by the NNSF of China with contract number 11671075.

**References**


