On One Method of Solution in Displacements Axisymmetric Problem
Theory of Elasticity in Spherical Coordinates for Radially Inhomogeneous Body

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Abstract. Various physical fields (temperature, radiation, humidity etc.) lead to continuous inhomogeneity of solids. If you have a point source, or source distributed on a spherical surface such fields have central symmetry. In elastic bodies, it leads to functional dependencies of the radius the mechanical characteristics of the material \(-E(r)\) and \(v(r)\). Replacement of the elastic constants on functions for solving elasticity problem leads to differential equations with variable coefficients. The paper deals with the numerical-analytical method of solving axisymmetric problem in spherical coordinates for radially inhomogeneous bodies. The method consists in reducing the system of equations in partial derivatives to an infinite system of ordinary differential equations of second order. The obtained system of ordinary differential equations is solved numerically.

Introduction

Calculations of constructions from inhomogeneous materials constitute a separate section of solid mechanics [1-3]. The author published papers dealing calculations of elastic and elastic-plastic inhomogeneous bodies [4 -7]. In this paper we present a method of separation of variables for radially inhomogeneous body in an axially symmetric problem in spherical coordinates.

State of the Problem

Axisymmetric problem of elasticity theory in spherical coordinates for radially inhomogeneous body based \(\frac{\partial}{\partial \phi} = 0\) and \(w = 0\) is reduced to two differential equations with respect to displacements:

\[
\mu \nabla^2 u + 3(\lambda + \mu) \frac{\partial \varepsilon_{av}}{\partial r} - \frac{2 \mu}{r^2} \left( u + \frac{\partial v}{\partial \theta} + v \cot \theta \right) + 3 \frac{\partial \lambda}{\partial r} \varepsilon_{av} + 2 \frac{\partial \mu}{\partial r} \frac{\partial u}{\partial r} - 3 \frac{\partial}{\partial r} \left( K \varepsilon_f \right) + R = 0;
\]

\[
\mu \nabla^2 v + \frac{3(\lambda + \mu)}{r} \frac{\partial \varepsilon_{av}}{\partial \theta} + \frac{\mu}{r^2} \left( 2 \frac{\partial u}{\partial \theta} - \frac{v}{\sin^2 \theta} \right) + \frac{\partial \mu}{\partial r} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) + \Theta = 0.
\]

Here \(\lambda, \mu\) and \(K\) are arbitrary functions of the radius, \(\varepsilon_{av}\) – average strain, \(\varepsilon_f\) – forced deformations (temperature, radiation, etc.), \(R\) and \(\Theta\) – volume forces.

\[
\varepsilon_{av} = (\varepsilon_r + \varepsilon_\theta + \varepsilon_\phi) / 3 = \frac{1}{3} \left( \frac{\partial u}{\partial r} + \frac{2 u}{r} + \frac{v}{r} \cot \theta \right)
\]

\[
\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right)
\]
Considering the spherical shape of body (solid or hollow), the boundary conditions for the stresses in axisymmetric problem can be written as follows:

\[ r = a, \quad \sigma_r = -p_a; \quad r = b, \quad \sigma_r = -p_b; \quad \tau_{r\theta} = q_a; \quad \tau_{r\theta} = q_b, \]

where \( a \) and \( b \) – the radii of the inner and outer surface of a thick-wall hollow sphere (in particular cases can be \( a = 0 \) and \( b \to \infty \)); \( p_a, p_b \) – normal, \( q_a, q_b \) – tangential surface loads.

The system of equations (1), (2) with the boundary conditions (3) represents a boundary value problem.

Using the Cauchy relation and Hooke's law, we can obtain expressions for the stresses:

\[
\sigma_r = \lambda \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\nu}{r} \cot \theta \right) + 2\mu \frac{\partial u}{\partial r} - 3K\epsilon_f; \\
\sigma_\theta = \lambda \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\nu}{r} \cot \theta \right) + 2\mu \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right) - 3K\epsilon_f; \\
\sigma_\phi = \lambda \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\nu}{r} \cot \theta \right) + 2\mu \left( \frac{u}{r} + \frac{\nu}{r} \cot \theta \right) - 3K\epsilon_f; \\
\tau_{r\theta} = \mu \left( \frac{\partial v}{\partial r} + \frac{\nu}{r} \cot \theta \right) 
\]

(4)

**Numerical-Analytical Method of Solving**

We seek a solution of equations (1), (2) in the form of expansions in Fourier series in Legendre polynomials:

\[
u(r, \theta) = \sum_{n=0}^{\infty} u_n(r) P_n(\cos \theta); \quad \nu(r, \theta) = \sum_{n=1}^{\infty} v_n(r) \frac{dP_n(\cos \theta)}{d\theta},
\]

(5)

where \( P_n(\cos \theta) \) – Legendre polynomial \( n \)-th degree which is a solution of equation [8, 9]:

\[
\frac{d^2 P_n(\cos \theta)}{d\theta^2} + \frac{dP_n(\cos \theta)}{d\theta} \cot \theta + n(n+1)P_n(\cos \theta) = 0.
\]

(6)

For integer values of \( n \) Legendre polynomials form a complete orthogonal system of functions in the range \( 0 \leq \theta \leq \pi \), so that

\[
\int_0^\pi P_n(\cos \theta) \cdot P_m(\cos \theta) d\theta = \begin{cases} 
0 & \text{at } n \neq m; \\
\frac{2}{2n+1} & \text{at } n = m.
\end{cases}
\]

From the theory of special functions [10] known that the expansion of in a Fourier series in Legendre polynomials has the same properties as any Fourier series, such as trigonometric functions.

Included in the equations (1), (2) volume loads and forced deformations also expanded in a series of type (5):

\[
R(\theta) = \sum_{n=0}^{\infty} R_n \cdot P_n(\cos \theta); \quad \Theta(\theta) = \sum_{n=1}^{\infty} T_n \cdot \frac{dP_n(\cos \theta)}{d\theta}
\]

(7)
where the expansion coefficients \( R_n, T_n \) are determined by the formulas:

\[
R_n = \frac{2n+1}{2} \int_{-1}^{1} R(t) \cdot P_n(t) \, dt; \quad T_n = \frac{1}{\left( \frac{dP_n(t)}{d\theta} \right)^2} \int_{-1}^{1} \Theta(t) \cdot \frac{dP_n(t)}{d\theta} \, dt; \quad (8)
\]

In (8) we use the notation \( t = \cos \theta \).

In solving the problem with a one-dimensional (radial) inhomogeneity, for example, when the dependence \( E(r) \) is due to centrally symmetric temperature field \( T = T(r) \), it is natural to consider the forced (temperature) deformations, also depending only on the radius. In this case, there is no need to lay \( \varepsilon_f \) in a series of Legendre polynomials.

To satisfy the boundary conditions (6) corresponding to an axisymmetric problem, should also be presented surface loads in the form of series:

\[
\begin{align*}
\left\{ p_a \right\} &= \sum_{n=0}^{\infty} \left( p_{a,n} \right) P_n(\cos \theta); \\
\left\{ q_a \right\} &= \sum_{n=1}^{\infty} \left( q_{a,n} \right) \frac{dP_n(\cos \theta)}{d\theta}.
\end{align*}
\]

The coefficients of expansion are determined by the formulas:

\[
\begin{align*}
\left\{ p_{a,n} \right\} &= \frac{2n+1}{2} \int_{-1}^{1} \left( p_a(t) \right) P_n(t) \, dt; \\
\left\{ q_{a,n} \right\} &= \frac{1}{\left( \frac{dP_n(t)}{d\theta} \right)^2} \int_{-1}^{1} \left( q_a(t) \right) \frac{dP_n(t)}{d\theta} \, dt. \quad (10)
\end{align*}
\]

Substituting (5) into (4) allows using equation (6) to receive representations of stresses included in the boundary conditions in the form of a series:

\[
\sigma_r = \sum_{n=0}^{\infty} \left[ (\lambda + 2\mu) u_r' + \frac{2\lambda}{r} u_n - \frac{\lambda n(n+1)}{r} v_n - 3K g_n \right] P_n(\cos \theta);
\]

\[
\tau_{\theta} = \sum_{n=1}^{\infty} \mu \left( v_r' - \frac{v_n}{r} + u_n \right) \frac{dP_n(\cos \theta)}{d\theta}. \quad (11)
\]

Here and below the prime denotes differentiation with respect to the radius. Using the relations (5) and (7) from equilibrium equations (1), (2) we obtain a number of systems (for each \( n \) of the two ordinary differential equations for the functions \( u_n(r) \) and \( v_n(r) \).

\[
\begin{align*}
(\lambda + 2\mu) u_n'' + & \left[ \frac{2(\lambda + 2\mu)}{r} + (\lambda + 2\mu) \right] u_n' + \\
+ & n(n+1) \left[ -\frac{\lambda + \mu}{r} v_n' + \left( \frac{\lambda + 3\mu}{r^2} - \frac{\lambda}{r} \right) v_n \right] - 3(Ke_f)' + R_n = 0; \quad (12)
\end{align*}
\]

\[
\begin{align*}
\mu v_n'' + & \left( \frac{2\mu}{r} + \mu \right) v_n' - \left[ n(n+1) \frac{\lambda + 2\mu}{r^2} + \frac{\mu'}{r} \right] v_n' + \frac{\lambda + \mu}{r} u_n' + \left[ \frac{2(\lambda + 2\mu)}{r^2} + \frac{\mu'}{r} \right] u_n + T_n = 0. \quad (13)
\end{align*}
\]

To equations (12) and (13) it is necessary to add boundary conditions for functions \( u_n(r) \) and \( v_n(r) \) that are written in the form:
\[ r = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} (\lambda + 2\mu)u_n' + \frac{2\lambda}{r}u_n - \frac{\lambda n(n+1)}{r}v_n - 3K\varepsilon_f \\ \mu \left( v_n' - \frac{v_n}{r} + \frac{u_n}{r} \right) \end{pmatrix} = \begin{pmatrix} -p_{a,n} \\ -p_{b,n} \end{pmatrix}; \quad (14) \]

So we got a number of problems (for each \( n \)), within the area described by the differential equations (12), (13), and on the surface - the relations (14). The choice of the number of members of the Fourier series should be decided on the basis of analysis of surface loads expansions in Fourier series in the formulas (9).

Given the arbitrary nature of the dependence of the mechanical characteristics of the material of the radius, a solution obtained by one-dimensional boundary value problems should be carried out numerically [11].

**Algorithm for the Numerical Solution**

In this chapter, for the numerical solution of one-dimensional boundary value problems similar to the method discussed in the [4, 7] two ordinary second-order differential equation (12), (13) for the functions \( u_n(r) \) and \( v_n(r) \) reduce to a system of four equations of the first order:

\[ \frac{dY_n}{dr} = A_n Y_n + F_n, \quad (15) \]

where \( Y_n \) – vector of unknown length of 4, wherein \( y_{1n} = u_n; \ y_{2n} = u_n'; \ y_{3n} = v_n; \ y_{4n} = v_n' \); \( A \) – matrix of the coefficients of the \( 4 \times 4 \), which coefficients are:

\[ a_{11} = 0; \ a_{12} = 1; \ a_{13} = a_{14} = 0; \quad a_{21} = \frac{n(n+1)\mu}{r^2(\lambda + 2\mu)} + \frac{2\lambda'}{r^2} + \frac{2\lambda'}{r(\lambda + 2\mu)}; \quad a_{22} = -\frac{2}{r} - \frac{\lambda' + 2\mu'}{\lambda + 2\mu}; \]

\[ a_{23} = -n(n+1)\left[ \frac{\lambda + 3\mu}{r^2(\lambda + 2\mu)} - \frac{\lambda'}{r(\lambda + 2\mu)} \right]; \quad a_{24} = +n(n+1)\frac{\lambda + \mu}{r(\lambda + 2\mu)}; \quad a_{31} = a_{32} = a_{33} = 0; \]

\[ a_{34} = 1; \ a_{41} = -\frac{2(\lambda + 2\mu)}{r^2\mu}; \ a_{42} = -\frac{\lambda + \mu}{r\mu}; \quad a_{43} = n(n+1)\frac{\lambda + 2\mu}{r^2\mu} + \frac{\mu'}{r\mu}; \quad a_{44} = -\frac{2}{r} - \frac{\mu'}{\mu}; \]

\( F_n \) – the vector of right-hand side of length 4, the components of which are expressed by the formulas: \( f_1 = f_3 = 0; \quad f_2 = \frac{3(Kg_n)'}{\lambda + 2\mu} - R_n; \quad f_4 = -\frac{T_n}{\mu} \).

The boundary conditions (14) can also be written in matrix form:

\[ r = a, b; \quad B_n Y_n = \Phi_n. \quad (16) \]

Here

\[ B_n = \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix}; \quad \Phi_n = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \]

In these equations are non-zero elements of the matrix \( B \) and the vector are defined by:

\[ b_{11} = \frac{2\lambda'}{r}; \ b_{12} = (\lambda + 2\mu); \ b_{13} = -\frac{\lambda n(n+1)}{r}; \quad b_{21} = \frac{\mu}{r}; \ b_{23} = -\frac{\mu}{r}; \ b_{24} = \mu; \]
Numerical solution of the problem described by equation (15) with boundary conditions (16) can be accomplished by matrix orthogonal shooting [11].

Summary
Given in the article the method of separation of variables in spherical coordinates for radially inhomogeneous bodies allows carrying out calculations of different designs (the protective shell in the thermal power construction, hemispherical domes, underground structures, etc.) in the presence of actions that change the mechanical properties of materials.

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References