Exact Solution of Yu-Toda-Sasa-Fukugama Potential Equation In (3+1) Dimension

Dongyu Cao, and Lifeng Guo*
Department of Mathematics, Northeast Petroleum University, Daqing, China

Abstract. This article solves the Yu-Toda-Sasa-Fukugama potential equation in (3+1) dimension [1-4]. First of all, we carry out the traveling wave transformation on the equation. Then, the equation can be transformed into integral form. At last, we use the complete discrimination system for polynomial [5-12] to obtain all exact solutions of the equation.

Keywords: Traveling wave transform, the YSFY potential equation in (3+1) dimension, Complete discrimination system for polynomial.

1 Introduction

Recent years, the study of nonlinear partial differential equations has attracted more and more mathematicians and physicists. Many scientists and researchers began to pay attention of exact solutions to nonlinear partial differential equations. For example, the elliptic function expansion method[13], the homogeneous balance method[14], the Riccati function method[15], the extending Riccati's mapping method[16] and so on. This paper studies the Yu-Toda-Sasa-Fukugama potential equation in (3+1) dimension, we will use the following method to find the exact solution of this equation.

2 The exact solution of the YTSY potential equation in (3+1) dimension

Let’s consider the Yu-Toda-Sasa-Fukugama potential equation in (3+1) dimension that has changed into a latent form,

$$-4u_{tt} + u_{xxxx} + 4u_t u_{xx} + 2u_x u_{xx} + 3u_{yy} = 0,$$

(1)

perform traveling wave transformation on this equation,

$$u(x, y, z, t) = u(\xi), \xi = ax + by + cz + dt,$$

(2)

where $a, b, c, d$ are constants. According to Eq.(1), we get
\[(3b^2 - 4ad)u'' + a^3 cu^{(4)} + 6a^2 cu'u'' = 0,\]  

(3)

integrating Eq. (3) once, we have

\[(3b^2 - 4ad)u' + a^3 cu''' + 3a^2 c(u')^2 = c_0,\]

(4)

where \(c_0\) is integral constant. Order \(v = u'\), we get

\[v'' = -2v^2 + \frac{4ad - 3b^2}{a^3 c} v + \frac{c_0}{a^3 c},\]

(5)

from Eq.(5), we get

\[(v')^2 = a_3 v^3 + a_2 v^2 + a_1 v + a_0,\]

(6)

where

\[a_3 = -\frac{2}{3},\]

\[a_2 = \frac{4ad - 3b^2}{a^3 c},\]

\[a_1 = \frac{c_0}{a^3 c},\]

\[a_0 = c_1,\]

(7)

where \(c_1\) is integral constant. Order \(x = (a_3)^\frac{1}{3} v, b_2 = a_2 (a_3)^\frac{2}{3}, b_1 = a_1 (a_3)^\frac{1}{3}, b_0 = a_0,\)

Eq.(6) can be rewritten as

\[(x')^2 = x^3 + b_2 x^2 + b_1 x + b_0,\]

(8)

write the Eq.(8) in integral form

\[\pm (a_3)^\frac{1}{3} (\xi - \xi_0) = \int \frac{dx}{\sqrt{F(x)}},\]

(9)

where

\[F(x) = x^3 + b_2 x^2 + b_1 x + b_0,\]

(10)

The Eq.(10) is third-order polynomial, the complete discrimination system for polynomial is
\[
\begin{align*}
\Delta &= -27\left(\frac{2b_2}{27} + b_0 - \frac{b_1 b_0}{3}\right)^2 - 4(b_1 - \frac{q_2}{3})^3, \\
D_1 &= b_1 - \frac{b_2}{3}
\end{align*}
\]

(11)

according to the Eq.(11), the classification of the solution can be obtained, there are four situations.

Case 1: If \( \Delta = 0, D_1 < 0 \). We have

\[F(x) = (x-r_1)^2(x-r_2),\]

(12)

where \( r_1, r_2 \) is real number and \( r_1 \neq r_2 \). If \( x > r_2 \), we get the solution of the Eq. (6)

\[
\begin{align*}
v_1 &= (a_3)^{-\frac{1}{3}} \left\{ (r_1 - r_2) \tanh^2 \left[ \frac{\sqrt{r_1 - r_2}}{2} \right] (a_3)^{\frac{1}{3}} (\xi - \xi_0) + r_2 \right\}, (r_1 > r_2) \\
v_2 &= (a_3)^{-\frac{1}{3}} \left\{ (r_1 - r_2) \coth^2 \left[ \frac{\sqrt{r_1 - r_2}}{2} \right] (a_3)^{\frac{1}{3}} (\xi - \xi_0) + r_2 \right\}, (r_1 > r_2), \\
v_3 &= (a_3)^{-\frac{1}{3}} \left\{ (r_2 - r_1) \tan^2 \left[ \frac{\sqrt{r_2 - r_1}}{2} \right] (a_3)^{\frac{1}{3}} (\xi - \xi_0) + r_2 \right\}, (r_1 < r_2)
\end{align*}
\]

(13)

where \( \xi_0 \) is integral constant.

Case 2: If \( \Delta = 0, D_1 = 0 \), we have

\[F(x) = (x-r_1)^3,\]

(14)

we get the solution of the Eq. (6)

\[v_4 = 4a_3^{-1}(\xi - \xi_0)^{-2} + a_3^{-\frac{1}{3}} r_1.\]

(15)

Case 3: If \( \Delta > 0, D_1 < 0 \). We have

\[F(x) = (x-r_1)(x-r_2)(x-r_3),\]

(16)

if \( r_1 < x < r_3 \), make variable substitution which is \( x = r_1 + (r_2 - r_1) \sin^2 \varphi \), we get the solution of the Eq. (6)

\[v_3 = (a_3)^{\frac{1}{3}} \left[ r_1 + (r_2 - r_1) \sin^2 \left( \frac{\sqrt{r_3 - r_1}}{2} \right) (a_3)^{\frac{1}{3}} (\xi - \xi_0), j \right],\]

(17)
if \( x > r_3 \), make variable substitution which is \( x = \frac{-r_2 \sin^2 \varphi + r_3}{\cos^2 \varphi} \), we get the solution of the Eq.(6)

\[
v_6 = (a_3)^{-\frac{1}{3}} \left[ r_3 - r_2 \text{sn}^2 \left( \frac{\sqrt{r_3 - r_1}}{2} (a_3)^{\frac{1}{3}} (\xi - \xi_0), j \right) \right]^{\frac{1}{3}} \left[ \text{cn}^2 \left( \frac{\sqrt{r_3 - r_1}}{2} (a_3)^{\frac{1}{3}} (\xi - \xi_0), j \right) \right]^{\frac{1}{3}},
\]

where \( j^2 = \frac{r_2 - r_1}{r_3 - r_1} \).

Case 4: If \( \Delta < 0 \). We have

\[
F(x) = (x - r_1)(x^2 + px + q),
\]

where \( p^2 - 4q < 0 \),

if \( x > r_1 \), make variable substitution which is \( x = r_1 + \sqrt{r_1^2 + pr_1 + q} \tan^2 \frac{\varphi}{2} \), we get the solution of the Eq.(6)

\[
v_7 = (a_3)^{-\frac{1}{3}} \left[ r_1 + \frac{2\sqrt{r_1^2 + pr_1 + q}}{1 + \frac{1}{r_1^2 + pr_1 + q}} \right] - \sqrt{r_1^2 + pr_1 + q},
\]

So far, this paper has obtained seven exact solutions of four types of the YTSF potential equation in (3+1) dimension. We can integrate the obtained \( v \) to obtain all the expressions of the original equation \( u \).

3 Conclusion

Firstly, this article transforms partial differential equations into ordinary differential equations through traveling wave transformation system for polynomial, all exact solutions of the equations are obtained. At last, we have got the solitary wave solutions, singular solutions and elliptic functions solutions. This method enriched the solution system of nonlinear partial differential equations and the solution process is easier, easier to understand and easy to apply.

Acknowledgments

This work was supported by the State Key Program of National Natural Science of China (Grant No.51834005), Guided innovation fund project of Northeast Petroleum University(Grant No.2020YDL-06); National Undergraduate Training Program for Innovation and Entrepreneurship(Grant No.202010220004); Innovation and entrepreneurship education course of Northeast Petroleum University(Grant No.CXCY-
References