Generalized Grouping Regularization for Time Domain Force Reconstruction Problems

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Abstract. Grouping regularization has received attention in recent studies because it offers flexibility and better noise immunity than traditional methods. In this paper, we propose the generalized grouping regularization (GGR). With different parameter settings, GGR provides localization ability, adaptive basis selection ability, and at the same time reconstruct forces in low signal-to-noise ratio (SNR) conditions. And with specific parameter settings, the proposed method degenerates to traditional Tikhonov regularization and compressed sensing. GGR is supposed to serve as the general regularization scheme for time domain force reconstruction problems. A cantilever beam simulation shows its superiority.

Introduction

In engineering practice, the knowledge of external loads is becoming crucial for design and health monitoring of structures. In cases where the external loads cannot be directly measured, the inverse approach is called for. The inverse approach means that, one measures the responses of structures at feasible locations and inversely calculate the external loads based on a known system model. However, the inverse approach is often ill-posed and regularization techniques are needed to stabilize its solution. The most widely adopted one is Tikhonov regularization\cite{1,2}. This technique adds the $l_2$ norm of the solution as a penalty to the reconstruction error functional. The solution to the new minimization problem will be a little biased but more immune to measurement noise.

Another recently widely discussed regularization technique is compressive sensing (CS)\cite{3,4}. CS theory basically tells that, if the sensing matrix and the basis matrix for the to-be-reconstructed signal is incoherent and the signal is sparse in the basis, we can reconstruct the signal with very few observations and overwhelming possibility for an accurate reconstruction. A CS reconstruction typically uses $l_1$ norm of the signal as the penalty. A force reconstruction version CS tells that if the force is sparse in a carefully chosen basis, a satisfactory reconstruction based on this basis can be achieved in low signal-to-noise(SNR) conditions.

The two above regularization treats each entry in the to-be-reconstructed force vector equally. Few researches consider grouping different entries\cite{5,6}. However, grouping, by incorporating possibilities, relaxes the requirement to one’s priori knowledge on the load condition before the actual reconstruction. One effect is that, it allows force localization from a few possible locations and at the same time, reconstruct the force history. Another effect is that it allows adaptive basis selection for the sparse force. This research will provide the generalized grouping regularization(GGR), and show its flexibility and superiority through a cantilever beam simulation.

Generalized grouping regularization

Suppose that the structure is time invariant and linear elastic. The system matrix $H$, i.e. Impulse Response Matrix (IRM) can be built with modal synthesis method,
Here $H_i$ denotes the $i$th-order IRM, $\phi_i$ is the $i$th-order mode shape, $s_r$ is the response measurement location, $s_f$ the force location, $\Delta t$ the sampling interval, $k$ the number of modes considered. If we denote the response measurement as a vector $x$, the naive force reconstruction will be the solution to the following problem,

$$ f = \arg \min_f \|Hf - x\|^2. $$

As we have mentioned, the above problem is ill-posed. This means that the matrix $H$ is usually rank-deficient or has large condition number, and a little amount of inevitable noise in $x$ will cause $f$ to change a great deal. In most cases, the solution to Eq.(2) is unusable. To stabilize the result, regularization techniques are called for. We directly provide the generalized grouping regularization (GGR), or $p$-$q$ norm regularization, and later we will show how this technique degenerates to classic ones,

$$ f = \arg \min_f \|Hf - x\|^2 + \alpha \|f\|_{p-q}. $$

$\alpha$ is called the regularization parameter. $\alpha$ is usually determined with a certain parameter selection criterion, such as $L$-curve, generalized cross validation, Bayesian information criterion, multiplicative regularization method and so on[7]. $f$ is evenly divided into $m$ groups and each group has $n$ items. Its $p$-$q$ norm is

$$ \|f\|_{p-q} = \left( \sum_{i=1}^{m} \left( \sum_{j=1}^{n} |f_{ij}|^p \right)^{q/p} \right)^{1/q}. $$

If $p$, $q$ are set different values, the GGR will have different effects.
1. $p = 2$, $q = 1$, $m = 1$. GGR degenerates to traditional Tikhonov regularization. This technique is most widely used and does not require specific nature of the force.
2. $p = 1$, $q = 1$, $m = 1$. GGR degenerates to classic CS. CS requires $f$ to be sparse-represented in the right basis.
3. $p = 2$, $q = 0.5$, $m > 1$. Group sparsity for localization. The force can be localized from a few possible locations, and reconstructed simultaneously.
4. $p = 1$, $0 < q < 1$, $m > 1$. Adaptive basis selection (ABS) method[8]. The CS requires the right sparse-representation basis for the force known a priori. ABS relaxes this requirement, and adaptively determines the appropriate basis for the force.

For different cases above, the solution to Eq.(3) requires different algorithms. The effects and algorithms will be illustrated by the following cantilever beam simulation.

The cantilever beam simulation

In this section, a cantilever steel beam simulation is used to demonstrate the above claims. For each case, the ability of GGR and its solution algorithm will be given. The beam is 0.4m long, with a cross section 0.03m x 0.01m. The beam is evenly meshed into 40 elements. The Young’s modulus and density are 1.76e5MPa and 7900kg/m$^3$ respectively. The beam model is shown in Fig.1.
Two sensors are used. Five potential force locations are considered.

Two displacement sensors are mounted 0.25m and 0.4m from the clamped end of the beam. The measurements of sensors are calculated by modal synthesis method in explicit format. 3 modes are taken into account. Five potential force locations, 0.08m apart from each other, are considered.

1) The Tikhonov regularization case. \( p = 2, q = 1, m = 1 \). In this case, the only one force location is considered and assumed known. The force is applied at the 3\textsuperscript{rd} position. Tikhonov regularization acts as the benchmark in most researches. It’s widely considered because it doesn’t require any specific nature of the force. So a force consisting multiple shapes is applied and 30-SNR Gaussian white noise is added to both the sensors. For Tikhonov regularization, a closed form solution exists,

\[
f = \left(H^T H + \alpha I \right)^{-1} H^T x.
\]

The Tikhonov solution and direct inverse one are compared in Fig.2.

2) The CS case. \( p = 1, q = 1, m = 1 \). In this case, the force location is also the 3\textsuperscript{rd} position. CS requires the force to be sparse-represented in a right basis, denoted by \( \mathbf{B} \). \( \mathbf{f} = \mathbf{B} \mathbf{y} \). The problem becomes

\[
y = \arg \min_{\mathbf{y}} \| \mathbf{H} \mathbf{y} - \mathbf{x} \|^2 + \alpha \| \mathbf{y} \|.
\]

\( \mathbf{y} \) is supposed to have few non-zero items. There are many algorithms for the solution to Eq.(6). Most takes advantage of the convexity of \( l_1 \) norm. Here, Sparse Reconstruction by Separable
Approximation (SpaRSA) method is adopted. 30-SNR noise is added to both the sensors. The results of $l_1$ and $l_2$ norm regularization are compared in Fig.3.

![Figure 3. CS result by SpaRSA method.](image)

It is seen that $l_1$ reconstruction estimates the impact peak better than Tikhonov regularization, since an impact is sparse in time domain.

3) The group sparsity localization case. $p = 2$, $q = 0.5$, $m = 5$. Five potential force locations are all considered to validate the localization ability. $\|f\|_{2,q,0.5}^2$ is still a convex norm, so a convexity-based Group-Least Absolute Shrinkage and Selection Operator (G-LASSO) is used. G-LASSO selects the optimal one among the five location groups. Since each group of entries stand for the force history at one potential location, G-LASSO is capable of both localizing and reconstructing the force. The result is shown in Fig.4. It can be seen that, after the localization of G-LASSO, only the components of the $3^{rd}$ group constitute a reasonable force history. The $3^{rd}$ location is determined as the true location. The force history is then reconstructed with classic Tikhonov regularization. In this case, 40-SNR noise is added to the sensor measurements.

![Figure 4. The G-LASSO localization and reconstruction result.](image)

The left subfigure: G-LASSO localization result. The entries of $3^{rd}$ potential location constitute a reasonable force history. The force history is better reconstructed with Tikhonov regularization considering only the $3^{rd}$ location.
4) The ABS case. ABS method regularizes with a non-convex $l_{p,q}$ norm. The initial solution for its iteration is important. Here a convenient and reliable algorithm, a special interior-point method is used for the initial solution. The method is basically a PCG-accelerated primal barrier method, and the gap between its functional and that of the dual problem is used as the stopping criterion. Then based on the initial solution, Iteratively Reweighted Least-Squares (IRLS) algorithm is applied for the final solution. The advantage of ABS method is that it allows considering multiple possible sparse-bases, and inducing sparsity on a group level. In this way, the priori knowledge on the sparse force is relaxed and better noise immunity is achieved. In this simulation, a basis matrix consisting of 4 bases is used,

$$ B = \begin{bmatrix} B_{\text{tri}} & B_{\text{Haar}} & B_{\delta} & B_{\cos} \end{bmatrix} \quad (7) $$

The bases are triangle basis, Haar wavelet basis, identity basis and discrete cosine basis respectively. $p = 1, q = 0.6$. 20-SNR noise is added to the sensor measurements. The ABS result is shown in Fig. 5. The adaptive basis selection ability is proved by the right subfigure. Only the coefficients in the triangle basis is selected. This means ABS method provides sparsity on the group level, other than on an entry level as for the naive CS method.

![Figure 5. The ABS result.](image)

The left subfigure: the reconstructed force history. The impact time and amplitude are well identified. The right subfigure: only the coefficients corresponding to the triangle basis remain non-zero. Though 4 bases are considered simultaneously, ABS method adaptively chooses the triangle basis to sparse-represent the impact.

Conclusions

In this research, we propose the generalized grouping regularization (GGR) for time domain force reconstruction problems. GGR contains a mixed $p$-$q$ norm of the force as a regularizer. By adjusting $p$ and $q$ to different set of values, GGR degenerates to classic methods, such as Tikhonov regularization, CS, Group-sparsity and ABS methods. To each case, GGR is given different characteristics and abilities, such as localization and adaptive basis selection. GGR serves as the general regularization scheme for time domain force reconstruction problems. And we hope this research inspires further investigation on parameter settings and reconstruction abilities of GGR in various contexts.
References


