An Exact Non-stationary Vortex Solution of the Plane Flow and Brownian Motion

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Based on the proposed conception "Pseudo-potential" of the incompressible plane flow, an exact non-stationary vortex solution of the Euler’s equation is given, \( f \) can be arbitrarily given potential force. It is proved that the given solution can describe the infinitely many unsteady vortices distributed periodically on the whole plane. The Melnikov function method is used to prove Brownian motion appearing along the border region separating different vortices.

Keywords: Non-stationary Vortex; Pseudo-potential; Euler’s Equation; Brownian Motion; Melnikov Function.

1. Introduction

The Brownian motion appears universally and that the Brownian motion becomes stronger and stronger as the temperature increases. Though some chaotic motion in the fluid convection can be explained through the Lorenz equation, which is derived from the Navier-Stokes equation \([1, 4, 6]\), but it had been long believed for us that the chaotic motion or the Brownian motion in the static flow might not be described with the Euler’s equation or with the Navier-Stokes equation since its only classical solution (which is the velocity distribution of the flow) should be zero everywhere for the static flow. The zero classical solution means that there is neither any macroscopic motion, nor the microscopic Brownian motion in the flow. We had thought that the microscopic Brownian motion in the static fluid could only be described by the statistical mechanics \([5, 7]\), or be described by the stochastic differential equation \([2, 3]\).

However, our above cognition will be subverted by an exact non-stationary vortex solution of the Euler’s equation

\[
\frac{\partial \psi}{\partial t} + (\psi \nabla) \psi = -\frac{1}{\rho} \nabla p + f
\]

which exact solution will be given in the this paper, based on the proposed conception "Pseudo-potential" of the incompressible plane flow. In fact, we will show the exact solution describes infinitely many unsteady vortices periodically...
distributed on the x-y plane, and the Brownian-like motion appears along the border region, which separates different vortices.

2. Pseudo-Potential flow

In the classical fluid mechanics [1, 6], an inviscid flow with zero vorticity is called potential. For potential flow in a simply connected plane region D, there is a scalar potential function \( \phi(x, y, t) \), such that for each t, the velocity field \( u \) of the fluid can be generated by \( u = \nabla \phi \). If the flow is incompressible, then the potential \( \phi \) is a harmonic function in \( x \in D \). Based on the relation between the harmonic function and the analytic function in complex domain, some important and interesting solutions of the Euler’s equation have been found.

The famous KdV equation for the waves on shallow water surfaces is derived from the Euler’s equation under the assistance of the velocity potential function [7]. In an attempt for deriving the Burgers-KdV equation from the Navier-Stokes equation, Yu and Guan posed the conception "pseudo-potential" for the viscous flow and the flow with vorticity. Though the Burgers-KdV equation has not been derived in our attempt, but the pseudo-potential has been proved to be useful in finding new exact solution [8].

A two dimensional flow is called a pseudo-potential flow, if its velocity \( v = (u(x, y, t), v(x, y, t)) \) can be generated by a scalar function \( \phi(x, y, t) \) according to the following formula.

\[
\left( u(x, y, t), v(x, y, t) \right) = \left( \frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y} \right).
\]  

(2)

From the Navier-Stokes equation and the Helmholtz equation, we see that the pseudo-potential \( \phi(x, y, t) \) must satisfy

\[
\frac{\partial^2 \phi}{\partial x \partial y} + \frac{i}{2} \left( \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} \right) = \nu \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right).
\]  

(3)

Besides Eq. (3), the pseudo-potential \( \phi(x, y, t) \) should satisfy

\[
\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = 0
\]  

(4)

to match the incompressible condition \( u_x + u_y = 0 \).

Eqs. (3) and Eq. (4) are just the equations satisfied by the pseudo-potential \( \phi(x, y, t) \) for a two dimensional incompressible pseudo-potential flow. In the next section, we will give a technique for finding the solutions to the equation system.
3. An Exact Non-Stationary Vortex Solution of the Euler’s Equation

3.1. A technique for finding exact solutions

Notice that the equation (4) has the following general solution

\[ \varphi(x, y, t) = f(\tau, t) + g(\theta, t), \quad \tau = x + y, \quad \theta = x - y \]  

(5)

where \( f(\tau, t) \) and \( g(\theta, t) \) are arbitrary functions.

Substituting Eq. (5) into Eq. (3), we find that \( f(\tau, t) \) and \( g(\theta, t) \) must satisfy

\[ \left( \frac{\partial^{3} f}{\partial \tau^{3}} - \frac{\partial^{3} y}{\partial \theta^{3}} \right) + 2 \left( \frac{\partial^{3} f}{\partial \tau^{3}} \frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \tau} \frac{\partial^{3} g}{\partial \theta^{3}} \right) - 2y \left( \frac{\partial^{3} f}{\partial \tau^{3}} - \frac{\partial^{3} y}{\partial \theta^{3}} \right) = 0. \]  

(6)

As a practical technique for finding exact solutions of Eq. (6), we may try some particular pairs of functions \( f(\tau, t) \) and \( g(\theta, t) \) such that the following term in Eq. (6) can be decoupled as a sum of two functions \( f(T, t) \) and \( g(\theta, t) \). Usually, this will lead to linear differential equations for the testing function \( f \) and \( g \). This technique may be called the decoupling of Eq. (6).

In fact, Yu have found a series exact solutions of Eq. (6) respectively by the decoupling technique [8]. In the next section, we will introduce an interesting solution by this technique.

3.2. An exact non-stationary vortex solution of the Euler’s equation

Based on Eq. (6), let us try the following pair of functions

\[ \{ f(\tau, t) = a_{1}(t) \tau + a_{2}(t) \sin(k\tau) + b_{1}(t) \cos(k\tau), \] \[ g(\theta, t) = d_{1}(t) \theta + d_{2}(t) \sin(k\theta) + b_{2}(t) \cos(k\theta) \} \]  

(7)

where \( k \) is an arbitrary given positive number, \( d_{1}(t) \) and \( d_{2}(t) \) are arbitrary given continuous function in \( t \), and \( a_{i}(t) \), \( b_{i}(t) \) \( (i = 1, 2) \) are functions to be determined the following formul:\n
\[ a_{1}(t) = A_{+} e^{-2k^{2}t^{2}} \cos(2k \int d_{1}(t) dt) \]  
\[ a_{2}(t) = A_{-} e^{-2k^{2}t^{2}} \cos(2k \int d_{1}(t) dt) \]  
\[ b_{1}(t) = -k^{2} A_{+} e^{-2k^{2}t^{2}} \sin(2k \int d_{1}(t) dt) \]  
\[ b_{2}(t) = -k^{2} A_{-} e^{-2k^{2}t^{2}} \sin(2k \int d_{1}(t) dt) \]

We will get an exact non-stationary vortex solution of the Euler’s equation, the generated components \( u(x, y, t), v(x, y, t) \) of the velocity as follows.
where \( A_1, A_2 \) are arbitrary given constants, and the potential \( U(x, y, t) \) of the external force \( f \) can be given arbitrarily.

4. Unsteady Vortices and Brownian Motion

4.1. Unsteady Vortices

In order to see the motion of the fluid particle of the flow Eq. (8), we should make a qualitative investigation to the related system of the ordinary differential equations. Change Eq. (8) to Eq. (9) as follows

\[
\begin{align*}
\frac{dx}{dt} &= k[A_2 \cos(2k \int d_3(t) \, dt - k(x + y)) + A_2 \cos(2k \int d_1(t) \, dt - k(x - y))] + d_1(t) + d_2(t) \\
\frac{dy}{dt} &= k[A_2 \cos(2k \int d_2(t) \, dt - k(x + y)) + A_1 \cos(2k \int d_2(t) \, dt - k(x - y))] - d_1(t) + d_2(t)
\end{align*}
\]  

where Eq. (9) is a Hamiltonian system. When \( A_1 = A_2 = 1 \), Eq. (9) becomes an autonomous Hamiltonian system. Figure 1 shows three different phase portraits of the system Eq. (9) in cases (a) \( k = A_1 = A_2 = 1 \), (b) \( k = 1, A_1 = 2, A_2 = 1 \) and (c) \( k = 1, A_1 = 1, A_2 = 2 \). It is proved that the given solution can describe the infinitely many unsteady vortices distributed periodically on the whole plane.
For the perturbed system Eq. (9), we assume further that, \( d_1(t) = e\sin(\omega t) \), \( d_2(t) = e\cos(\omega t) \), and choose the corresponding integral constants \( c_1 = c_2 = 0 \) for the indefinite integrals \( \int d_1(t) \, dt = -\frac{e}{\omega} \cos(\omega t) + c_1 \) and \( \int d_2(t) \, dt = \frac{e}{\omega} \sin(\omega t) + c_2 \). Now the Hamiltonian system (9) can be written concretely as

\[
\begin{align*}
\frac{dx}{dt} &= Ak[\cos(k(x + y)) - 2ek\sin(\omega t) + \cos(k(x - y)) + 2ek\cos(\omega t)] \\
&\quad + e\sin(\omega t) + e\cos(\omega t) \\
\frac{dy}{dt} &= Ak[\cos(k(x - y)) + 2ek\cos(\omega t) - \cos(k(x + y)) - 2ek\sin(\omega t)] \\
&\quad - e\sin(\omega t) + e\cos(\omega t)
\end{align*}
\]

(10)

In case of \( k > 0, A_1 = A_2 = A > 0 \), the Melnikov function is

\[
M_2(t_0) = n\left[ \frac{\pi^2}{\omega} \operatorname{csch}\left( \frac{\pi(\omega t_0)}{2A^2} \right) - \frac{(4\omega^2 t_0^2 + \omega^2)}{4\pi \cdot \omega^2} \operatorname{sech}\left( \frac{\pi(\omega t_0)}{2A^2} \right) \right] \cos(2\omega t_0) (11)
\]

which has also infinitely many simple zeros at \( t_0 = \left( n \pm \frac{1}{2} \right) \frac{\pi}{\omega}, n = 0, \pm 1, \pm 2, \ldots \). By that results, we may say that the motion of a fluid particle, which starts at the initial time \( t_0 \), is Brownian-like.

### 4.2. Brownian motion

The Brownian-like motions of the fluid particles may appear in a region larger. The numerical tests show that, most of motions of the fluid particles starting from a small neighborhood of any given heteroclinic orbit at the initial time \( t_0 \), are Brownian-like. Based on the numerical results under the parameter assignment \( A = k = \omega = 1, e = 0.01 \), Figure 2 shows four Brownian-like motion loci in the time interval \( 0 \leq t \leq 3000 \), where the case (i) is the locus from the initial position \( (x(0), y(0)) = (0, -0.13) \), (ii) is the locus from \( (0, -0.1299) \), (iii) the locus from \( (0, 0) \), (iv) the locus from \( (0, 0.0001) \). Their initial points are at a small neighborhood of the center of the heteroclinic orbit. By comparing the differences of the initial points between (i) and (ii), between (iii) and (iv). We can see the sensible dependence of the motion behavior on its initial position and the Brownian-like motion appears along the border region, which separates different vortices.
(i) \((x(0), y(0)) = (-0.13, 0.13)\)

(ii) \((x(0), y(0)) = (-0.1299, 0.13)\)

(iii) \((x(0), y(0)) = (-0.13, 0.03)\)

(iv) \((x(0), y(0)) = (0.0001, 0.0001)\)

(v) Experimental phenomena

Figure 2. Some Brownian Motion and Comparison with experimental phenomena.
5. Conclusions

Though the solution Eq. (9) may provide us a possibility of a description of the practical two dimensional static fluid with temperature and with some complex motions inside, we must emphasize here that this paper does not wish to show that the Euler’s equation can provide a perfect and intrinsic description for the complex motion of the incompressible and inviscid fluid. More profound work in both physics and mathematics are still in need.

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References