Has the Function Relationship of the Four Diagonal Matrix Inverse Eigenvalue Problem of the Algorithm

Yun-fei WANG, Zhi-bin LI
School of Mathematics and Physics, Dalian Jiaotong University,
Dalian, Liaoning, China
E-mail: wangyunfeiky@163.com, lizhibinky@163.com

Matrix eigenvalue problems frequently encountered in many fields of mathematics and science and technology, is one of the core subject of numerical algebra, its research has important theoretical significance and application value. Our study is determined by eigenvalue and eigenvector of the matrix elements, known as the matrix inverse eigenvalue problem. The problem is ill-posed, so need to add some restrictive conditions. Therefore, matrix inverse eigenvalue problem is under certain conditions, according to the information to determine matrix eigenvalue and eigenvector of the elements. In this paper, we study on the condition of a given eigenvalue and eigenvector down backward four diagonal matrix problem. This paper discusses the existence and uniqueness of the solution, given the solution algorithm, and provides a numerical example.

Keywords: Four Diagonal Matrices; Inverse Eigenvalue Problem; Algorithm.

1. Bring Forward the Question

Matrix inverse eigenvalue problem is under certain conditions, how to determined matrix elements according to the information of eigenvalue and eigenvector. Matrix eigenvalue problems in quantum chemistry, quantum mechanics, mechanics, structural design, pattern recognition, particle physics, nuclear spectroscopy, the pole assignment of linear multivariable control system, etc. many fields has important application [2].

This paper studies the four diagonal matrices with the function relation refers to the following the n order matrix in the form of:

---

*This work is supported by the National Natural Science Foundation of China under Grant No.61273022.
In it, \( a_i \in R (i = 1, 2, \cdots, n), b_j, c_i \in R, c = f (b_j) (i = 1, 2, \cdots, n - 1) \),
\( d_i \in R, d \), \( d = f (b_j) (i = 1, 2, \cdots, n - 3) \),
\( f (x) (x \in R), \phi_i (x) (x \in R) \) are real function [1].

This paper presents the following inverse eigenvalue problem for matrices:

**Question FDMIEP**: Given two vary real number \( \lambda, \mu \) and non-zero vector
\( x = (x_1, x_2, \cdots, x_n)^T \in R^n \), \( y = (y_1, y_2, \cdots, y_n)^T \in R^n \), Struct \( J \) to make
\( (\lambda, x), (\mu, y) \) are the characteristic pairs of \( J \).

Record

\[
\begin{align*}
x_{i-1} &= x_0 = x_{n+1} = y_{i-1} = y_0 = y_{n+1} = 0 \\
D_{ij} &= \begin{bmatrix} x_i & y_i \\ x_{i-2} & y_{i-2} \end{bmatrix} \ (i = 3, 4, \cdots, n) \\
E_{ij} &= \begin{bmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{bmatrix} \ (i = 1, 2, \cdots, n - 1)
\end{align*}
\]

2. **The solution of problem**

Because \( (\lambda, x) \) is the characteristics of the four diagonal matrix \( J \), so

\[
\begin{align*}
a_1x_1 + b_1x_2 &= \lambda x_1, \quad (5-1) \\
c_1x_1 + a_2x_2 + b_2x_3 &= \lambda x_2, \quad (5-2) \\
d_1x_1 + c_2x_2 + a_3x_3 + b_3x_4 &= \lambda x_3, \\
\cdots & \cdots & \cdots \\
& \cdots & \cdots \\
d_{n-4}x_{n-4} + c_{n-3}x_{n-3} + a_{n-2}x_{n-2} + b_{n-2}x_{n-1} &= \lambda x_{n-2}, \quad (5-n-2) \\
d_{n-3}x_{n-3} + c_{n-2}x_{n-2} + a_{n-1}x_{n-1} + b_{n-1}x_n &= \lambda x_{n-1}, \quad (5-n-1)
\end{align*}
\]
\[ d_{n-2}x_{n-2} + c_{n-1}x_{n-1} + a_nx_n = \lambda x_n . \]  \hspace{1cm} (5-n) 

Because \((\mu, y)\) is the characteristics of the four diagonal matrix \(J\), so

\[ a_1y_1 + b_1y_2 = \mu y_1, \]  \hspace{1cm} (6-1) 
\[ c_1y_1 + a_2y_2 + b_2y_3 = \mu y_2, \]  \hspace{1cm} (6-2) 
\[ d_1y_1 + c_2y_2 + a_3y_3 + b_3y_4 = \mu y_3, \]  \hspace{1cm} (6-3) 

\[
\begin{align*}
&d_{n-4}y_{n-4} + c_{n-3}y_{n-3} + a_{n-2}y_{n-2} + b_{n-2}y_{n-1} = \mu y_{n-2}, \\
&d_{n-3}y_{n-3} + c_{n-2}y_{n-2} + a_{n-1}y_{n-1} + b_{n-1}y_{n} = \mu y_{n-1}, \\
&d_{n-2}y_{n-2} + c_{n-1}y_{n-1} + a_ny_n = \mu y_n.
\end{align*}
\]  \hspace{1cm} (6-n)

Inversing \(b_i, c_i (i = 1, 2, \cdots, n-1), d_i (i = 1, 2, \cdots, n-2)\).

Because of (5-1),(5-2)and(6-1),(6-2), we can get

\[ b_1E_i = (\mu - \lambda)x_1y_1, b_2E_2 = (\mu - \lambda)x_2y_2 + c_1E_1. \]  \hspace{1cm} (7)

Because of (5-n)and(6-n), we can get

\[ c_{n-1}E_{n-1} = d_{n-2}D_n - (\mu - \lambda)x_ny_n. \]  \hspace{1cm} (8)

By (5)and (6), we can get

\[
\begin{align*}
&d_{i-2}x_{i-2} + c_{i-1}x_{i-1} + a_ix_i + b_ix_{i+1} = \lambda x_i \quad (i = 3, 4, \cdots, n-1) \\
&d_{i-2}y_{i-2} + c_{i-1}y_{i-1} + a_iy_i + b_iy_{i+1} = \mu y_i \quad (i = 3, 4, \cdots, n-1)
\end{align*}
\]  \hspace{1cm} (9) \hspace{1cm} (10)

Then we can get

\[ d_{i-2}D_i - c_{i-1}E_{i-1} + b_iE_i = (\mu - \lambda)x_iy_i \quad (i = 3, 4, \cdots, n-1). \]  \hspace{1cm} (11)

Because \(c_i = f_i(b_i)(i = 1, 2, \cdots, n-2), d_i = g_i(b_i)(i = 1, 2, \cdots, n-3)\), If \(E_i \neq 0(i = 1, 2, \cdots, n-1)\). Because of (7), (8)and (11), we can get

\[ b_i, c_i (i = 1, 2, \cdots, n-1), d_i (i = 1, 2, \cdots, n-2) \].

Above all, for the \textbf{FDMIEP}, we can get
Theorem If \( E_i \neq 0 (i = 1, 2, \cdots, n-1), D_n \neq 0 \), so the FDMIEP has the only solution \([3]\), and

\[
\begin{align*}
    b_1 &= \frac{(\mu-\lambda)x_1y_1}{E_1}, \\
    b_2 &= \frac{(\mu-\lambda)x_2y_2 + c_iE_i}{E_2}, \\
    b_i &= \frac{(\mu-\lambda)x_iy_i - d_{i-2}D_i + c_{i-1}E_{i-1}}{E_i} \quad (i = 3, 4, \cdots, n-1), \\
    c_i &= f_i(b_i)(i = 1, 2, \cdots, n-1), \quad d_i = \varphi_i(b_i)(i = 1, 2, \cdots, n-3), \quad d_{n-2} = \frac{(\mu-\lambda)x_ny_n + f_{n-1}(b_{n-1})E_{n-1}}{D_n}.
\end{align*}
\]

3. Algorithm

(1) initialization: import \( n, \lambda, \mu, x, y \)

\[ f_i(x)(i = 1, 2, \cdots, n-3). \]

(2) estimate: towards \( (i = 1, 2, \cdots, n-1) \), count \( E_i \), If \( E_i = 0(1 \leq i \leq n-1) \), Problem without a solution. when \( E_i \neq 0 \), count \( D_n \), If \( D_n = 0 \), Problem without a solution \([5]\).

(3) count: towards \( (i = 1, 2, \cdots, n-1) \), count

\[
\begin{align*}
    b_i &= \frac{(\mu-\lambda)x_iy_i - d_{i-2}D_i + c_{i-1}E_{i-1}}{E_i} \quad \text{towards} \ (i = 1, 2, \cdots, n-1), \quad c_i = f_i(b_i); \\
    d_i &= \varphi_i(b_i), \quad \text{towards} \ (i = 1, 2, \cdots, n-3), \quad d_{n-2} = \frac{(\mu-\lambda)x_ny_n + c_{n-1}E_{n-1}}{D_n} \quad \text{towards} \ (i = 1, 2, \cdots, n), \quad \text{count} \ a_i.
\end{align*}
\]

372
If $x_i \neq 0$, $a_i = \lambda - \frac{d_{i-1}x_{i-2} + c_{i-1}x_{i-1} + b_i x_{i-1}}{x_i}$;
If $y_i \neq 0$, $a_i = \mu - \frac{d_{i-1}y_{i-2} + c_{i-1}y_{i-1} + b_i y_{i-1}}{y_i}$ [4].

4. Numerical Example

Example 1

Let $\lambda = 1$, $\mu = 2$, $x = (1, 0, 1, 0, -1)^T$, $y = (1, 2, 3, 4)^T$,

$$f_i(x) = 2x, f'_i(x) = x, f'_i(x) = x^2, \varphi_i(x) = x.$$ 

It is easy to calculate $E_1 = 2 \neq 0$, $E_2 = -2 \neq 0$, $E_3 = 7 \neq 0$,

$D_4 = -2 \neq 0$, so the FDMIEP has the only solution.

And $b_1 = \frac{(\mu - \lambda)x_1y_1}{E_1} = \frac{1}{2}$, $c_1 = f'_i(b_1) = 2 \times \frac{1}{2} = 1$,

$$d_1 = \varphi_i(b_1) = \frac{1}{2},$$

$$b_2 = \frac{(\mu - \lambda)x_2y_2 + c_1E_1}{E_2} = \frac{1 \times 2 - 2}{-2} = -1,$$

$$D_3 = x_3y_1 - y_3x_1 = 1 - 3 = -2,$$

$$b_3 = \frac{(\mu - \lambda)x_3y_3 + c_2E_2 - d_1D_3}{E_3} = \frac{3 + (-2) - (-2) \times \frac{1}{2}}{2} = \frac{2}{7},$$

$$c_2 = 1, d_2 = \frac{(\mu - \lambda)x_4y_4 + f'_i(b_3)E_3}{D_4} = \frac{12}{7}, c_3 = \frac{4}{49},$$

$$a_1 = \lambda - \frac{b_2x_2}{y_2} = 1 - 0 = 1,$$

$$a_2 = \mu - \frac{c_3y_1 + b_3y_3}{y_2} = 2 - \frac{1 + 3 \times (-1)}{2} = 3,$$
\[ a_3 = \lambda - \frac{d_3x_1 + c_2x_2 + b_1x_3}{x_3} = 1 - \frac{1}{2} + \frac{(-1) \times 2}{7} = \frac{11}{14}, \]

\[ a_4 = \lambda - \frac{d_3x_2 + c_1x_3 + b_1x_4}{x_4} = 1 - \frac{4}{49} \cdot \frac{53}{49}. \]

So

\[
J = \begin{pmatrix}
1 & \frac{1}{2} & 0 & 0 \\
1 & 3 & -1 & 0 \\
\frac{1}{2} & 1 & \frac{11}{14} & \frac{2}{7} \\
0 & \frac{12}{7} & \frac{4}{49} & \frac{53}{49}
\end{pmatrix}
\]

And \( Jx = \lambda x, Jy = \mu y \).

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant No.61273022.

References


