The Method of Matrix Differential Equation to Calculate the Natural Frequencies of Variable Cross-section Beams on the Non-uniform Elastic Foundations

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Abstract. The tradition methods to calculate the natural frequencies of beam on the elastic foundation were based on high order differential equation by defection. This paper derived the general equation based on the one order matrix differential equation represented by the mixed variable (a vector formed by deflection, angle of rotation, bending moment and shearing force) to calculate the natural frequencies of the variable cross-section beams on the non-uniform elastic foundations. To calculate the natural frequencies of this kind of beams, we divide the beams into several sections, calculate the transfer matrices of each section, multiply the matrices in proper order and obtain the overall transfer matrix of the mixed variable from one end to the other end of the variable cross-section beam on non-uniform elastic foundations. According to the conditions of boundary at both ends of the beam, we can calculate out the natural frequencies in MATLAB easily.

Introduction

The traditional methods to calculate the natural frequencies of beams on the elastic foundation were commonly based on the high order differential equation represented by deflections[1,2,3]. The solving processes of traditional methods are complicated. Otherwise, traditional methods frequently supposed that foundations are uniform, which are usually not corresponding with the reality. In practical projects, sometimes the foundation beams may stride across several kinds of foundations and may be variable cross-sections[4,5,6,7]. This paper devotes to solve this kind of difficult problems in traditional methods by a simple and unify way.

Different from traditional methods, in this paper the general equation based on the one-order matrix differential equation represented by the mixed variable (a vector formed by deflection, angle of rotation, bending moment and shearing force) is derived to calculate the natural frequencies of the variable cross-section beams on the non-uniform elastic foundations. The transfer matrix of the mixed variables from one end to the other end of the foundation beam is given by solving the matrix differential equation. The transfer matrix has a unified and simple form. According to the conditions of boundary at both ends of beam, the natural frequencies can be calculated out easily.

The method in this paper can be programming in MATLAB easily.

General Equation

A homogeneous and constant cross-section beam on the uniform elastic foundation is shown in Figure 1. Its length is \( L \).

![Figure 1. A homogeneous and constant cross-section beam on uniform elastic foundation.](image-url)
Set the origin of coordinate system at the left end of the beam. The positive direction of deflection $y$, angle of rotation $\theta$, bending moment $M$ and shearing force $Q$ are shown in Figure 1.

Let $y_t$, $\theta_t$, $M_t$, and $Q_t$ represent respectively the deflection, angle of rotation, bending moment and shearing force at $x$ cross-section of beam at time $t$. They must meet the following differential relations:

$$\frac{\partial y_t}{\partial x} = \theta_t, \quad \frac{\partial \theta_t}{\partial x} = M_t, \quad \frac{\partial M_t}{\partial x} = Q_t, \quad \frac{\partial Q_t}{\partial x} = -\bar{m} \frac{\partial^2 y_t}{\partial t^2} - ky_t,$$

(1)

In which $EI$ represents the flexural rigidity of the beam, $\bar{m}$ represents the mass per unit length of the beam, $k$ represents the bedding value of elastic foundation.

Let

$$y_t = ye^{i\omega t}, \quad \theta_t = \theta e^{i\omega t}, \quad M_t = Me^{i\omega t}, \quad Q_t = Qe^{i\omega t}$$

(2)

In which, $\omega$ represent the natural frequencies of transverse vibration of beam.

Substituting Eq. 2 into Eq. 1, we obtain

$$\frac{dy}{dx} = \theta, \quad \frac{d\theta}{dx} = \frac{M}{EI}, \quad \frac{dM}{dx} = Q, \quad \frac{dQ}{dx} = (\bar{m}\omega^2 - k)y$$

(3)

Convert Eq. 3 into matrix form

$$\frac{d}{dx} \begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \bar{m}\omega^2 - k & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \theta \\ M \\ Q \end{bmatrix}$$

(4)

Introduce symbol

$$R = [y \quad \theta \quad M \quad Q]^T$$

In which $R$ is called mixed variable, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \bar{m}\omega^2 - k & 0 & 0 & 0 \end{bmatrix}$$

(5)

Eq. 4 can be rewrote into the differential equation of the mixed variable

$$\frac{dR}{dx} = AR,$$

Solving this differential Eq, we obtain

$$R = e^{Ax}R^L$$

(6)

In which $R^L$ represents the mixed variable at the left end of beam. Substituting $x = l$ into Eq. 6, we can obtain the relation between the mixed variable at the right end and at the left end:

$$R^R = e^{Al}R^L$$

(7)

Introducing symbol $T = e^{Al}$, Eq. 6 can be rewrote into the following form:
\( R^R = TR^L \)

or

\[
\begin{bmatrix}
  y^R \\
  \theta^R \\
  M^R \\
  Q^R
\end{bmatrix} =
\begin{bmatrix}
  t_{11} & t_{12} & t_{13} & t_{14} \\
  t_{21} & t_{22} & t_{23} & t_{24} \\
  t_{31} & t_{32} & t_{33} & t_{34} \\
  t_{41} & t_{42} & t_{43} & t_{44}
\end{bmatrix}
\begin{bmatrix}
  y^L \\
  \theta^L \\
  M^L \\
  Q^L
\end{bmatrix}
\]

The elements \( t_{ij} \) in Eq. 9 can be obtained through expanding \( e^{Al} \) into Taylor series[8,9]

\[
T = e^{Al} = I + Al + \frac{1}{2!} A^2 l^2 + \frac{1}{3!} A^3 l^3 + \cdots + \frac{1}{n!} A^n l^n + \cdots
\]

In which, \( I \) is the unit matrix. Above series is convergent for arbitrary \( A \). The resolution of \( t_{ij} \) can be obtained through using Eq. 10.

Let

\[
S_p = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(4n + P)!}, \quad \lambda^2 = \frac{(\bar{m} \omega^2 - k)l^4}{EI} \quad (P = 0, 1, 2, 3)
\]

Then

\[
T = e^{Al} = IS_p + AlS_1 + A^2 l^2 S_2 + A^3 l^3 S_3
\]

The corresponding matrix coefficients are listed as below

\[
t_{11} = t_{22} = t_{33} = t_{44} = S_0, \quad t_{12} = t_{34} = IS_1, t_{21} = t_{43} = \frac{\lambda^2}{l}S_3
\]

\[
t_{13} = t_{24} = l^2 EI S_2, t_{31} = t_{42} = (\bar{m} \omega^2 - k)l^2 S_2, t_{14} = \frac{l^3}{EI} S_3
\]

\[
t_{41} = (\bar{m} \omega^2 - k)lS_1, t_{23} = \frac{l}{EI} S_1, t_{32} = (\bar{m} \omega^2 - k)l^3 S_3
\]

Substituting the boundary conditions into Eq. 9, the natural frequencies of transverse vibration of beam can be obtained.

Example. A beam that two ends simply supported is on the elastic foundation. Its length is \( l \). Its mass per unit length is \( \bar{m} \). Its flexural rigidity is \( EI \). The whole length is supported by the uniform elastic foundation. The bedding value of the elastic foundation is \( k \).

The boundary foundations are

\[ x = 0, \quad y^L = 0, \quad M^L = 0; \quad x = l, \quad y^R = 0, \quad M^R = 0 \]

Substituting these into Eq. (9), we get

\[
t_{13} \theta^L + t_{14} Q^L = 0, \quad t_{32} \theta^L + t_{34} Q^L = 0
\]

Since \( \theta_0 \) and \( Q_0 \) are not equal to zero

\[
\Delta(\lambda) = \begin{vmatrix}
  t_{12} & t_{14} \\
  t_{32} & t_{34}
\end{vmatrix} = 0
\]

Using expansion Eq. 13, the high order algebra equation can be obtained

\[
\frac{1}{\lambda^2}[(\lambda + \frac{\lambda^5}{5!} + \frac{\lambda^9}{9!} + \cdots)^2 - (\frac{\lambda^3}{3!} + \frac{\lambda^7}{7!} + \frac{\lambda^{11}}{11!} + \cdots)^2] = 0
\]
Solving Eq. (15), we can obtain

\[
\frac{1}{\lambda^3} \left( \lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \frac{\lambda^7}{7!} + \cdots \right) \sin \lambda = 0
\]  

From Eq. (16), we know

\[
\sin \lambda = 0 \quad \lambda = n\pi \quad (n = 1, 2, 3, \cdots)
\]

Substituting the solution into Eq. 11, the natural frequencies of transverse vibration of the beam that two ends are simply supported on the uniform elastic foundation can be obtained

\[
\omega = \sqrt{\frac{EI}{m_i}} \pi^4 (n^4 + \frac{k^2}{EI \pi^4}) \quad (n = 1, 2, 3, \cdots)
\]

**Calculate the Natural Frequencies of the Variable Cross-Section Beams on the Non-Uniform Elastic Foundations**

A step shape beam on the elastic foundation can be divided into \( n \) sections. Each section has its corresponding flexural rigidity \( EI_i \), its corresponding mass per unit length \( m_i \). Each elastic foundation has its corresponding bedding value \( k_j \). Two sections of a step shape beam on non-uniform elastic foundation are shown in Figure 2. The right section with constant cross-section strides across two kinds of foundations. So we divide the beam into three sections: \( i \), \( i+1 \) and \( i+2 \).

![Figure 2. Two sections of a step shape beam on non-uniform elastic foundation.](image)

The transitive relation of the mixed variable between the left end and right end of the section \( i \) is:

\[
R_i^R = T_i R_i^L
\]  

(17)

In which \( T_i \) is the transfer matrix of the section \( i \).

The transitive relation of mixed variable between the both ends of the section \( i+1 \) that is adjacent to the section \( i \) is:

\[
R_{i+1}^R = T_{i+1} R_{i+1}^L
\]  

(18)

According to the continuity condition of the connecting section

\[
R_{i+1}^L = R_i^R
\]  

(19)

We can obtain

\[
R_{i+1}^R = T_{i+1} T_i R_i^L
\]  

(20)

Likewise, we can obtain
\[ R^R_{i+2} = T_{i+2} T_{i+1} T_i R^L_i \]  

Analogously, we can obtain the transfer equation of the step shape beam on elastic foundation of the mixed variables from the left end to the right end.

\[ R^R_n = T_n T_{n-1} \cdots T_1 T_1 = TR^L_1 \]  

In which \( T \) is called accumulation matrix of the step shape beam on elastic foundation.

\[ T = T_n T_{n-1} \cdots T_2 T_1 \]  

Substituting boundary conditions into Eq. 22, we can calculate natural frequencies of any kind of step shape beam on non-uniform elastic foundation.

We can calculate the natural frequencies of continuously varying section beam through dividing it into \( n \) constant micro-section and then using the above method.

**Conclusion**

In this paper the general equation based on the one order matrix differential equation represented by the mixed variable is derived to calculate the natural frequencies of transverse vibration of variable cross-section beams on the non-uniform elastic foundations. And it can be used easily.

**References**


