Phonic Localization in Locally Resonant Ternary Thue-Morse and Rudin-Shapiro Structures

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ABSTRACT: The wave localization properties in locally resonant ternary phononic aperiodic crystals based on Thue-Morse and Rudin-Shapiro sequences, are theoretically investigated by the transfer matrix method. The various factors affecting localization characteristics are discussed. Numerical results show that the effect of material properties and the geometric structure parameters on the number and size of band gaps is obvious. Moreover, compared with the corresponding binary aperiodic structures without local resonances, the localization factors become larger in the locally resonant ternary structures. And wider and lower-frequency band gaps and localized resonant flat bands appear in the locally resonant aperiodic system. Compared with the locally resonant Thue-Morse system, the localization factor becomes larger and the band splitting phenomenon is less obvious in the locally resonant Rudin-Shapiro system, which is just the opposite to that in the aperiodic system without local resonance. This study is useful in filter, waveguide, and so on.

KEYWORDS: Aperiodic phononic crystal; Local resonance; Transfer matrix method; Wave localization.

1 INTRODUCTION

After decades of intense studies focused on periodic phononic crystals (PNCs) known as the counterpart of photonic crystals [1], in view of their potential capability of controlling the acoustic/elastic wave propagation [2], scientific interest is also growing in aperiodic phononic crystals (APNCs), which are characterized by a lack of long-range periodic translational order [3-8]. Such structures are capable of exhibiting anomalous properties of great interest for both basic and applied science. For example, they should not have propagation characteristics of Bloch waves, but exhibit unique characteristics of the mixture of acoustic/elastic wave propagation and localization. These peculiar properties are useful to the fabrication of the acoustic or elastic wave filters, the design of transducers, noise control, and so on. To date, significant research efforts have been made to investigate the APNCs. Sesion et al. [4] considered the acoustic-phonon transmission spectra in Fibonacci superlattices. King et al. [5] reported experimental observation of phononic band structure in one-dimensional aperiodic waveguide structures. Aynaou et al. [6] studied the propagation and localization of acoustic waves in Fibonacci phononic circuits. Parsons et al. [7] observed longitudinal acoustic phonons in porous silicon superlattices. Gazi et al. [8] experimentally studied the transmittance of longitudinal acoustic waves in one-dimensional Fibonacci hypersonic phononic crystals of porous silicon. In these previous studies, only longitudinal wave propagating normally was considered. However, compared to the above studies, coupling of longitudinal and transversal elastic waves propagating obliquely in APNCs has received less attention. Moreover, most of the previous investigations on wave propagation in APNCs focused on the Bragg’s reflection mechanism. This may limit the potential applications of such materials. In addition, previous studies on aperiodic phononic crystals are mainly focused on different arrangement of two components, to our best knowledge, the investigations on three-component APNCs are limited.

In an ongoing series of recent investigations [9, 10], we have studied the localization properties of layered phononic crystals. In particular, in [9], we showed the localization properties in randomly disordered periodic phononic crystals (PNCs) with and without local resonances. Subsequently, in [10], we explored the localization properties of aperiodic phononic crystals. However, a comprehensive and detailed study of the localization taking account into the aperiodicity and local resonance, simultaneously, has not been reported so far.

In this paper, different from the previously reported works, we present comparative study of localization properties of the locally resonant ternary Thue-Morse and Rudin-Shapiro structures. The
localization factor is calculated through the transfer matrix method. The chapter is structured as follows: Section 2 introduces the localization factor based on the transfer matrix method. Section 3 is devoted to the illustration and discussion of representative results from the localization factor calculations concerning the different aperiodic arrangements with local resonances, and finally in Section 4, we address some brief conclusions and perspectives.

2 THEORETICAL METHOD

In this paper, we investigate from a theoretical point of view the phononic properties of elastic waves with coupling of longitudinal and transverse modes obliquely propagating in an aperiodic layered media. We employed the transfer matrix method [9] in order to study the localization properties in aperiodic structures based on the accurate calculation of their localization factor. The APNCs under study is constructed from the Thue-Morse (TM) and Rudin-Shapiro (RS) sequences. TM sequence is generated by replacing ‘H’ with ‘HL’ and replacing ‘L’ with ‘LH’. According to these inflation rules, the lower orders TM sequence is H, HL, HLLH, HLLHLHHL, etc. and the inflation rule used to generate the Rudin-Shapiro arrays can simply be obtained by the iteration of the two-letter inflation as follows: H → HHL, HLL → HHLHHLHHL, where H and L stand for the sub-layers made of different materials, respectively.

Let a wave be incident at an angle \(0^\circ \leq \theta \leq 90^\circ\) onto a multilayer which is composed of H and L stacked alternately following the TM and RS sequence. The well-known governing equation of the wave motion can be written as

\[
\rho \ddot{U}_m(X,t) = \sum_{m} \left[ \sum_{m} \partial_{m} \left( \dot{U}_m(X,t) \right) + \partial_{m} \ddot{U}_m(X,t) \right],
\]

(1)

where \(\partial_{m}\) and \(\partial_{i}\) are partial derivative with respect to the \(i\)th position vector \(X\) component and time, respectively, \(\rho\) is mass density, and \(\kappa, \nu\) are Lamé coefficients. For a harmonic elastic wave of angular frequency \(\omega\), Eq. (1) reduces to the time-independent form,

\[
\sum_{m} \left[ \sum_{m} \partial_{m} \left( \dot{U}_m(X,t) \right) + \partial_{m} \ddot{U}_m(X,t) \right] = -\rho \omega^2 U_m(X,t).
\]

(2)

Introducing two scalar potentials \(\xi\) and \(\eta\), for a typical alternating segment \(m\) with material \(m\) and length \(L_m\) arrayed along the \(x\) dimension, considering the Snell’s Law, the general solutions can be expressed as

\[
\xi_m(t_m, \nu_m, t) = [A e^{-i\kappa_m \tau_m} + B e^{i\kappa_m \tau_m}] e^{(i\nu_m - i\kappa_m \tau_m)}
\]

\[
\eta_m(t_m, \nu_m, t) = [C e^{-i\kappa_m \tau_m} + D e^{i\kappa_m \tau_m}] e^{(i\nu_m - i\kappa_m \tau_m)}
\]

(3)

where \(A, B, C\), and \(D\) are unknown state parameters, \(\tau_m = x_m / L_m\), and \(\nu_m = y_m / L_m\) is the dimensionless local coordinates, \(L_m\) is the mean value of the thickness of material \(H\), \(L^2 = -1\), \(\omega\) is the circular frequency, \(\kappa_m\) is the dimensionless wave vector component along the \(y\)-axis;

\[
\tau_{lm} = \sqrt{\left(\omega L_m / c_m\right)^2 - \kappa_m^2}, \quad \tau_{lm} = \sqrt{\left(\omega L_m / c_m\right)^2 - \kappa_m^2}.
\]

Applying the continuous conditions at the interface of segment \(m\) and \(m+1\) in the \(st\) cell give

\[
M_m V_m^{(s)} = T_{m+1} V_{m+1}^{(s)}
\]

(4)

where \(V_m^{(s)} = [A_m, B_m, C_m, D_m]^T\). So, the continuity conditions at the interface of the \(st\) and \(s+1\)th cells give

\[
M_n V_n^{(s)} = T_{n+1} V_{n+1}^{(s)}
\]

(5)

Substituting \(V_n^{(s)} = T_{n+1} M_{n+1}^{-1} T_{n+1} M_{n+2}^{-1} \cdots T_{n+3} M_{n+4}^{-1} M_{n+5} V_1\) into (5) leads to

\[
V_{s+1}^{(s+1)} = T_{s+1} M_{s+1}^{-1} T_{s+1} M_{s+2}^{-1} \cdots T_{n+3} M_{n+4}^{-1} M_{n+5} V_1
\]

(6)

where the total transfer matrix \(T_{s+1} M_{s+1}^{-1} T_{s+1} M_{s+2}^{-1} \cdots T_{n+3} M_{n+4}^{-1} M_{n+5}\) is obtained. The detailed mathematical process will not be given here because they are lengthy. The reader may refer to many publications [9-10]. If the dimension of the transfer matrices is \(2p \times 2p\), then the smallest positive Lyapunov exponent \(\chi_p\) is the localization factor. The expression was given as follows:

\[
\chi_p = \lim_{n \to \infty} \frac{1}{n} \sum_{s=1}^{n} \ln \left\| \psi_{s+1}^{(p)} \right\|
\]

(7)

where the vector \(\psi_{s+1}^{(p)}\) is obtained through iteration and Gram-Schmidt orthonormalization procedures.

3 NUMERICAL RESULTS AND DISCUSSIONS

In this part, the locally resonant ternary aperiodic structures considered here are built by inserting the third material (soft rubber) between the two adjacent sublayers of the corresponding binary aperiodic systems. Detailed calculation will be performed for ternary APNCs with local resonances. The physical
parameters of materials involved in calculation are listed in Table.

Table Material constants of Rubber, Al, PP, Pb and Epoxy.

<table>
<thead>
<tr>
<th>medium</th>
<th>mass density $\rho$ (kg/m$^3$)</th>
<th>velocity of longitudinal wave $c_l$ (m/s)</th>
<th>velocity of transversal wave $c_t$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rubber</td>
<td>1300</td>
<td>22.87</td>
<td>5.547</td>
</tr>
<tr>
<td>Aluminum(Al)</td>
<td>2700</td>
<td>6410</td>
<td>3110</td>
</tr>
<tr>
<td>Polypropylene(PP)</td>
<td>910</td>
<td>1830</td>
<td>880</td>
</tr>
<tr>
<td>Plumbum</td>
<td>11400</td>
<td>2160</td>
<td>860</td>
</tr>
<tr>
<td>Epoxy</td>
<td>1200</td>
<td>2830</td>
<td>1160</td>
</tr>
</tbody>
</table>

3.1 *Thue-Morse locally resonant PNCs with different material combinations*

First, to compare the differences between the structures with and without local resonances in aperiodic systems, we consider the normal propagation of in-plane waves through an aperiodic ternary layered composite consisting of Pb-Rubber-Epoxy-Rubber layers based on the Thue-Morse sequence. The thickness ratio of Pb, Epoxy and Rubber is taken as $L_1 = 2L_2 = 100L_3$. By introducing $\Omega_{l1} = \omega L_1 / c_{l1}$, Fig.1 presents the localization factor versus the normalized frequency. It can be seen that compared with the aperiodic system without local resonances, the localization factor becomes larger in the locally resonant structure. And wider and lower-frequency band gaps appear in the locally resonant system. For example, Fig. 1 shows a low frequency band-gap (0.01182, 3.33) in the ternary composite structure with local resonance. In Fig.2, the localization factors versus the $\Omega_{l1}$ and the incident angle $\theta$ are illustrated in the grey-scale map for the two systems shown in Fig. 1(a) and (b). Compared with Fig. 2(a), band gaps and pass bands are less clear for the locally resonant aperiodic system. From Fig. 2 (b), it is interesting to note the presence of many flat bands crossing the complete Brillouin’s zone. This phenomenon is very similar to that observed in reference [9]. However, the number of flat bands is more than that of reference [9]. In addition, it can be seen that the localization factor is not nearly zero at very low frequencies no matter how the incidence angle varies, which is different from that in the aperiodic system without local resonance.

![Figure 1](image1.png)

![Figure 2](image2.png)

In the following, we discuss the influences of the material properties on the localization of the locally resonant ternary Thue-Morse sequence. We choose six material combinations: Pb-Rubber-Epoxy-Rubber, Pb-Rubber-Al-Rubber, Pb-Rubber-PP-Rubber, Epoxy-Rubber-Al-Rubber, Epoxy-Rubber-PP-Rubber and Al-Rubber-PP-Rubber. The localization factors versus the normalized frequency are shown in Fig. 3. It can be seen that the number and size of band gaps are different for the six systems. Results show that the material properties have obvious influences on band gaps of the APNCs.
3.2 Rudin-Shapiro locally resonant PNCs with different material combinations

To investigate the influences of material properties on the localization in locally resonant Rudin-Shapiro sequence, the results for the above six systems are shown in Fig. 4, respectively. The phenomenon that the number and size of band gaps are different for the six APNCs as shown in Fig. 3 can also be observed in Fig. 4. Compared with the Thue-Morse system (Fig. 3), the localization factor becomes larger and the band splitting phenomenon is less obvious in the locally resonant Rudin-Shapiro system, which is opposite to that in the aperiodic system without local resonance [10]. In addition, the comparison results of Fig. 3 and Fig. 4 show that the structural parameter is also influential for localization factors of APNCs, which means that the locally resonant ternary Thue-Morse sequence is the best candidate for the band splitting function with the weakest wave attenuation.

4 CONCLUSIONS

The results presented in this paper are based on numerical calculations localization properties of the locally resonant ternary Thue-Morse and Rudin-Shapiro phononic crystals by using the transfer matrix method. The conclusions are drawn as follows:

(i) The effect of material properties and the geometric structure parameters on the number and size of band gaps is obvious.

(ii) Compared with the corresponding binary aperiodic structures without local resonances, the localization factors become larger in the locally resonant ternary structures. And wider and lower-frequency band gaps and localized resonant flat bands appear in the locally resonant aperiodic system.

(iii) Compared with the locally resonant Thue-Morse system, the localization factor becomes larger and the band splitting phenomenon is less obvious in the locally resonant Rudin-Shapiro system, which is just the opposite to that in the aperiodic system without local resonance.

5 ACKNOWLEDGEMENTS

Support by the National Natural Science Foundation of China (No. 11002026, 11372039), Beijing Natural Science Foundation (No. 3133039), the Scientific Research Foundation for the Returned (No. 20121832001) are gratefully acknowledged.

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