Equivalent Stress Tensor and Anisotropic Failure Criterion of Soils

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ABSTRACT: Experimental evidence shows that the strength of geomaterials is significantly influenced by the inherent anisotropy, the intermediate principal stress and the stress direction. To measure the joint effect of the fabric tensor and the stress tensor, a second-order tensor named the equivalent stress tensor is extracted from the equilibrium equation of the soil-skeleton. As the magnitude and direction of the effective stress together with the microscopic characteristics of anisotropy are enrolled in the equivalent stress tensor, the modeling process of anisotropic failure criterion with the application of the equivalent stress tensor is proposed. On this basis, a general 3D failure criterion of anisotropic geomaterials is formulated based on the SMP criterion, of which all introduced parameters can be conveniently determined by conventional laboratory tests. Comparison with experimental results of various stress paths from literatures claims that the anisotropic failure criterion is able to capture the strength anisotropy in torsional shear tests. Further discussion is done on the possible application of the equivalent stress and the probable improvement of the new criterion to address other models.

INTRODUCTION

In general, the principal stresses or the stress invariants are applied in the analysis of the mechanical properties of materials, of which only three degrees of freedom are taken into consideration. This approach of neglecting stress directions is based on the hypothesis that the material is isotropic and the change of stress direction has no effect on the mechanical property of materials. However, it has been proved that the change of the orientation of the principal stress axes has an obvious influence on soil’s deformation and strength even if magnitudes of the principal stresses are kept constant (Miura et al. 1986, Pietruszczak and Mroz 2001, Gao et al. 2010, Lade et al. 2014, Aghajani and Salehzadeh 2015).

Anisotropy is one of the fundamental properties of geomaterials, which brings great difficulties in modeling. Traditionally, within the framework of plasticity, nonlinear empirical models, kinematic hardening models and boundary surface models are established. Although these anisotropic models have a high precision, most of them concern on the description of the macro-phenomenon while ignore the microcosmic mechanism of anisotropy. Microanalysis of granular materials indicates that the joint effect of the fabric and the stress are the essence of the anisotropic mechanical behavior. Therefore, the projection of stress on fabric (Pietruszczak and Mroz 2001, Lade 2007, 2008, Gao et al. 2010, Kong et al. 2013, Rodriguez and Lade 2013) and the joint invariants of the fabric and the stress (Li and Dafalias 2002, Lashkari 2014, Gao and Zhao 2015) are used to modify the yield function with an added anisotropic
strength parameters. However, additional assumptions are needed in these two approaches of modelling. Besides, Tobita and Yanagisawa (1988) and Oda (1993) made an attempt on modifying the stress of anisotropic materials. In these studies, the modified stress tensor, which is the product of the stress tensor and the fabric tensor, was directly used in describing the yielding characteristics of anisotropic granular materials. Lacking of theoretical basis, this method has not attracted much attention.

Based on the micro-meso-macro analysis of granular materials, this paper proposed a mechanical model of anisotropic soil, called the skeleton-cell model. Following an idea similar to that used in Oda (1993), the equivalent stress, which is acting on the soil-skeleton, is derived out from the equilibrium equation. With the application of the equivalent stress, a formulation of a general 3D failure criterion for cross-anisotropic soils for both nonrotating and rotating stresses is present here. Comparison with experimental results from literatures shows that the new anisotropic failure criterion is able to capture the strength anisotropy of soils.

EQUIVALENT STRESS TENSOR

Mechanical Model

The saturated soil is a typical two-phase material. In meso-scale, the soil-skeleton, which is composed of interconnected soil particles, can bear and pass the effective stress. Soil particles and water are assumed to be incompressible. The principle of effective stress indicates that the deformation and strength of soil are the deformation and strength of soil-skeleton. Thus, compared to the Cauchy stress, we tend to pay more attention to the effective stress of the soil-skeleton.

The anisotropy of geomaterials can be attributed to the difference in effective bearing area of the soil-skeleton in different directions, based on the assumption that the effective bearing area and the mechanical properties of each soil-skeleton are the same. As shown in Figure 1 (a), anisotropic geomaterials can be simplified into a reticular structure of two groups of orthogonal skeletons in 2-dimensions problem. Therefore, the distribution is uniform in each group, while there are differences among different groups. In addition to the soil-skeleton, the liquid phase constitutes a series of interconnected water-cells which is shown in Figure 1 (b). The combination of soil-skeletons and water-cells constitutes the skeleton-cell model, of which the stress acting on the parallel skeleton is the same and the pore water pressure is equal everywhere.

Taking a hexahedral element, of which the volume is 1 and the porosity is \( n \), as an example, the volume of the soil-skeleton and the water-cell are \((1-n)\) and \( n \) respectively. In the skeleton-cell model, the anisotropy of soil is reflected through differences of the ratio between the effective bearing area of soil-skeleton \( S_{s} \) and the effective bearing area of water-cell \( S_{w} \) on different directions. Defining the ratio of void area as \( \frac{S_{w}}{S_{s}} \), the following relationship can be obtained.

\[
S_{w} = (\bar{n}_{w} - \bar{n}_{s}) S
\]

\[
\int \bar{n}_{w} \, dx = \int \frac{S_{w}}{S} dx = \frac{V_{w}}{V} = n
\]

Based on the Cauchy stress theory and the average stress theorem, the equilibrium equation of soil-skeleton is

\[
(1 - \bar{n}_{w,i}) \bar{\sigma}_{w,j} + (1 - \bar{n}_{s,i}) \rho' g_i + f'_{w,j} = 0
\]
The equilibrium equation of water-cell is
\[ \bar{\eta}_{q,j} \bar{u}_{q,j}^w + \bar{\eta}_{q,j} \rho^w g_i - f_{q,j}^m = 0 \] (4)

Finally, the macroeconomic equilibrium equation for anisotropic materials can be written as
\[ (1-\bar{n}_{q,j}) \bar{\sigma}_{q,j} + \bar{\eta}_{q,j} \bar{u}_{q,j}^w + \bar{\eta}_{q,j} \rho^w g_i = 0 \] (5)

where, \( \bar{n} \rho^m g \), \( (1-\bar{n}) \rho^w g \) and \( \bar{\eta} \rho^w g \) are the gravity of saturated soils, soil-skeleton and water-cell respectively. \( f^m \) and \( f^w \) are the acting force and reaction force between the skeleton and water. \( \bar{\sigma} \) is the effective stress acting on the isotropic soil-skeleton and \( \bar{u}_w \) is the pore pressure of water-cells.

The analysis of skeleton-cell model shows that stress of macro-scale is distributed again in meso-scale. It should be noted that the pore pressures of anisotropic soils are equal over the element, which means that \( \bar{u}_w = u_w \), while the effective stresses are not. The equilibrium equation of isotropic soil is
\[ (1-n) \sigma_{q,j} + nu_{q,j}^w + n \rho^w g_i = 0 \] (6)

The relationship between the effective stress in macro-scale and the stress of isotropic soil-skeleton can be expressed by
\[ (1-n) \sigma_{ij} = (\delta_{ij} - \bar{n}_{ij}) \bar{\sigma}_{ij} \] (7)

Therefore, \( \bar{\sigma} \) reflects the real stress of soil-skeletons, which is named as the equivalent stress in order to facilitate the distinction with the effective stress \( \sigma \).

![Figure 1. Stress analysis of skeleton-cell model (a) Soil-skeleton. (b) Water cell Fabric Tensor.](image)

Since the stress induced anisotropy is extremely complicated, only the inherent anisotropy of soils is discussed in this paper. Firstly proposed by Brewer, the fabric tensor \( F \), has become a popular option in quantifying the degree and orientation of the inherent anisotropy of materials. In general, \( F \) is a symmetric second-order tensor, of which the trace is unity (Oda 1993). Magnitudes of components of the fabric tensor depend on the chosen of the coordinate system. Therefore, \( F \) owns three principal components called the principal fabrics of which the corresponding coordinate system is the principal fabric coordinate system. The following fabric tensor is adopted for the description of the cross-anisotropy (Gao 2010) of which the bedding plane are horizontal and the principal fabric coordinates system are concentric with the coordinates of the physical space.
\[ F = \begin{bmatrix} F_1 & 0 & 0 \\ 0 & F_2 & 0 \\ 0 & 0 & F_3 \end{bmatrix} = \frac{1}{3+\Delta} \begin{bmatrix} 1-\Delta & 0 & 0 \\ 0 & 1+\Delta & 0 \\ 0 & 0 & 1+\Delta \end{bmatrix} \] (8)

where \( \Delta \) is a scalar that characterizes the magnitude of the cross-anisotropy and makes the fabric tensor possess a unit trace. Its value is zero when the material is isotropic. Based on the characters of micro-structure, several kinds of fabric
tensor with different definition have been proposed. In this paper, the contact tensor suggested by Oda (1993) on the basis of the vector area $s$ of the contacting surface between particles is adopted to measure the degree of anisotropy of granular materials. As the area ratio of void $\tilde{n}_y$ is defined to describe the distribution of the solid-liquid phase of the material, the fabric tensor can be obtained by the statistical value of the effective bearing area of the soil-skeleton by

$$\delta_y - \tilde{n}_y = (1 - n) F_y \tag{9}$$

The effective stress $\sigma$ is defined in the macro-scale with the soil-skeleton as the research subject, while the equivalent stress $\tilde{\sigma}$ is the stress applied on the isotropic soil-skeleton in meso-scale. It should be noted that, no matter what kind of definition is used, the specific value of the fabric tensor cannot be directly measured, especially in engineering, so does the area ratio of void. Thus, the fabric tensor can only be obtained through the macroscopic physical quantity indirectly.

**Equivalent Stress Tensor**

As mentioned above, the joint effect of fabric and stress should be taken into consideration for anisotropic materials. Based on the mechanism analysis of skeleton-cell model, Eq. (7) can be rewritten by Eq. (9), as

$$\tilde{\sigma}_y = \sigma_y F_y^{-1}/3 \tag{10}$$

where $F_y^{-1}$ is the inverse of the fabric tensor $F_y$ ($F_y^{-1} F_y = \delta_y$). To ensure that $\tilde{\sigma}_y$ and $\sigma_y$ are the same for an isotropic material, $\Delta = 0$ and $F_y = \delta_y / 3$, the constant 3 is introduced in Eq. (10). Similar approaches was adopted by Oda (1993) and Li and Dafalias (2002). Moreover, the equivalent stress in Eq. (10) is derived out based on the assumption that base vectors of the fabric tensor are coaxial with the effective stress tensor. In the torsional shear test, the bedding plane of the specimen is often oriented horizontally while the radial stress is the intermediate principal stress. In the case that the axes of the major and minor principal stress rotate around the intermediate principal stress by an angle of $\alpha$, the conversion matrix $M$ can be expressed as follows

$$M_y(\alpha) = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix} \tag{11}$$

Conversion matrix in the form of Eq. (11) defines a basic rotation by an angle $\alpha$ around the $y$ axis. Similar matrices of rotation, whose rotational angles are $\beta$, $\gamma$ about axes $z$, $x$, can be written as

$$M_z(\beta) = \begin{bmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{bmatrix} \tag{12}$$

Therefore, the product

$$M = M_y(\alpha)M_z(\beta)M_x(\gamma) \tag{13}$$

represents the conversion matrix with combined rotations of three directions. Eq. (10) is obtained on the assumption that the basic vectors of $F$ and $\sigma$ are coaxial. Thus, in a more general case, the conversion matrix $M$ is needed in transferring the fabric tensor to the principle stress coordinate system and the following relationship can be obtained

$$\tilde{\sigma}_y = M_{m} M_{m} \sigma_m F_y^{-1}/3 \tag{14}$$
The conversion matrix $M$ reflects the transformation between base vectors of $F$ and $\sigma$. It can also be regarded as the projection of the principal fabric $F_0$ in the principal stress space. Therefore, the basic vectors of the equivalent stress determined by Eq. (14) are similar with the fabric tensor. The coordinate system of the principal stress is usually coincide with the principal fabric space, which means that $Q_0 = \delta_0$ and Eq. (14) can be simplified to Eq. (10). For isotropic materials, as components of fabric tensor follows $F_1 = F_2 = F_3 = 1/3$, the equivalent stress expressed by Eq. (10) further degenerates to the effective stress.

### ANISOTROPIC FAILURE CRITERION

#### Theoretical Framework

In general, the strength criterion of isotropic soils can be expressed as

$$F(\sigma) = 0$$  \hspace{1cm} (15)

The mechanical property of soil depends on the soil-skeleton. Thus, Eq. (15) describes the failure of isotropic soil-skeleton with the effective stress. Such isotropic failure criterion cannot be applied to anisotropic materials because the macro-mechanical characters of the two materials are different. However, the soil-skeleton can be regarded as a kind of isotropic material on the mechanical, which has been well-measured by isotropic materials. Taking the soil-skeleton as research object, the equivalent stress $\bar{\sigma}$ provides a direct approach of characterizing both the orientation and intensity of the inherent material anisotropy. As a result, the failure criterion in the form of the equivalent stress can be used to describe the strength of soil-skeleton by the following equation

$$F(\bar{\sigma}) = 0$$  \hspace{1cm} (16)

The microscopic characteristics of anisotropic soils are enrolled in the equivalent stress which also reflects the magnitude and direction of the effective stress. Therefore, it is convenient to take the isotropic soil-skeleton in micro-scale as the research subject in measuring the macro-mechanical characters of anisotropic materials with effective stress.

### Anisotropic SMP criterion

Taking the SMP criterion given by Matsuoka and Nakai (1974) as an example, the isotropic SMP criterion is expressed as

$$F(\sigma) = \sqrt{\frac{I_1 I_3 - 9 I_2}{9 I_1}} = \text{const.}$$  \hspace{1cm} (17)

where, for frictional materials the fixed value is $\tan \varphi$. Similar with the definition of the stress invariants, $I_1$, $I_2$ and $I_3$, the first, second and third equivalent stress invariants, $\bar{I}_1$, $\bar{I}_2$, and $\bar{I}_3$, can be obtained by

$$\bar{I}_1 = \text{tr}(\bar{\sigma}) = \bar{\sigma}_1 + \bar{\sigma}_2 + \bar{\sigma}_3$$  \hspace{1cm} (18)

$$\bar{I}_2 = \frac{1}{2} \left[ \text{tr}(\bar{\sigma})^2 - \text{tr}(\bar{\sigma}^2) \right] = \bar{\sigma}_1 \bar{\sigma}_2 + \bar{\sigma}_2 \bar{\sigma}_3 + \bar{\sigma}_3 \bar{\sigma}_1$$  \hspace{1cm} (19)

$$\bar{I}_3 = \det(\bar{\sigma}) = \bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}_3$$  \hspace{1cm} (20)

As has been mentioned above, the soil-skeleton can be regarded as an isotropic material. We propose the following expression to describe the failure behavior of the soil-skeleton.

$$F(\bar{\sigma}) = \sqrt{\frac{I_1 I_3 - 9 I_2}{9 I_3}} = \text{const.}$$  \hspace{1cm} (21)
The anisotropic yield function, which is expressed by Eq. (21), is called the ESMP shorted from the equivalent SMP failure criterion. As ESMP defines the failure behavior of the soil-skeleton, new parameters are need in order to distinguish ESMP from SMP. For frictional materials, the equivalent internal friction angle \( \phi \) of soil-skeleton is introduced to take place of the internal friction angle \( \varphi \) in Eq. (17). The anisotropic yield criterion adopting \( \bar{\sigma} \) can also be written in the form of \( \sigma \) with Eq. (14). Thus, ESMP can be regarded as an anisotropic failure criterion expressed by the effective stress.

**Calibration of model parameter**

With the application of the equivalent stress tensor, the key problem of establishing an anisotropic failure criterion becomes to the determination of the fabric tensor of materials, strength parameters of isotropic soil-skeleton and rotational angles between \( F \) and \( \sigma \). For the sake of simplicity, only the initial anisotropy is taken into account. This simplification assumed that the magnitude and direction of stress have no effect on the fabric or \( \Delta \). However, \( \Delta \) cannot be measured directly as mentioned above. In laboratory, three groups of tests can be used to determine \( \Delta \) and \( \phi \), including true triaxial shear tests on horizontal specimen at \( \theta_\sigma = 0^\circ \) and \( 120^\circ \) (see Figure 2a), triaxial tests on horizontal and vertical specimen (see Figure 2b) and torsion shear tests on horizontal specimen at \( \alpha = 0^\circ \) and \( 90^\circ \) (see Figure 2c). Therefore, the direction of \( \sigma_i \) can be well controlled at \( \alpha = 0^\circ \) and \( \alpha = 90^\circ \) respectively by each group of tests. In three dimension, the internal friction angle is defined by

\[
\varphi = \arcsin \frac{\sigma_i - \sigma_{\alpha}}{\sigma_i + \sigma_{\alpha}}
\]

Equation (22)

The shear strength \( \varphi \) of anisotropic soils depends heavily on the direction of \( \sigma_1 \), while the equivalent internal friction angle \( \phi \) of the soil-skeleton can be regarded as isotropic. The connection between these two research object is the equivalent stress. When \( \alpha = 0^\circ \), \( Q_j = \delta_j \), we have

\[
\sigma_1 = \bar{\sigma}_1 F_1, \quad \sigma_2 = \bar{\sigma}_2 F_2, \quad \sigma_3 = \bar{\sigma}_3 F_3
\]

Equation (23)

Strength of anisotropic soils at this condition can be expressed as

\[
\left( \frac{\sigma_1}{\sigma_3} \right)_{\alpha=0^\circ} = \frac{1 + \sin \phi_0}{1 - \sin \phi_0} = k_0, \quad \frac{\bar{\sigma}_1 F_1}{\bar{\sigma}_3 F_3} = k_0
\]

Equation (24)
where $k_0$ is a strength index. The subscript 0 indicates that the physical quantity is measured at $\alpha = 0^\circ$. When it comes to the soil-skeleton, the shear strength becomes

$$\tilde{\sigma}_i = \frac{1 + \sin \tilde{\phi}}{1 - \sin \tilde{\phi}} = \tilde{k}$$ (25)

where, $\tilde{k}$ is an equivalent strength index of the soil-skeleton. According to Eq.(24) and Eq. (25), we arrive at

$$F_i/F_i = k_0/\tilde{k}$$ (26)

Similarly, when $\alpha = 90^\circ$, $Q_{ij} = \delta_{ij}$. Relationship between $\sigma_i$ and $\tilde{\sigma}_i$ turns to

$$\frac{\sigma_i}{\tilde{\sigma}_i}_{\alpha=90^\circ} = \frac{1 + \sin \phi_0}{1 - \sin \phi_0} = k_0$$ (27)

Using the subscript $90$ to express the physical quantities measured at $\alpha = 90^\circ$, the strength of anisotropic soils can be expressed as following

$$\frac{\sigma_i}{\tilde{\sigma}_i}_{\alpha=90^\circ} = \frac{1 + \sin \phi_0}{1 - \sin \phi_0} = k_0, \quad \frac{\tilde{\sigma}_i F_i}{\sigma_i F_i} = k_0$$ (28)

As the soil-skeleton is isotropic, the shear strength can also be expressed as Eq. (25). Therefore, with Eq. (25), (27) and (28), we have

$$F_i/F_i = \tilde{k}/k_0$$ (29)

Finally, the following relationship between the components of fabric tensor can be obtained

$$F_i/F_i = \sqrt{k_0/k_0}$$ (30)

The parameter $\Delta$ is calculated by

$$\Delta = \frac{1 - \sqrt{k_0/k_0}}{1 + \sqrt{k_0/k_0}}$$ (31)

Strength parameters of the soil-skeleton can be expressed as

$$\tilde{k} = F_3/k_0, \quad \tilde{\phi} = \arcsin \frac{\tilde{k} - 1}{\tilde{k} + 1}$$ (32)

With above-mentioned method, the fabric tensor can be measured as long as the test data with different rotational angles of the conversion matrix are plenty, even if the material is anisotropic in three directions or the principal fabric and the principal stress are not coaxial.

**COMPARISON WITH EXPERIMENTAL RESULTS**

A total of 44 drained torsion shear tests have been carried out on fine Nevada sand by Lade et al. (2014) at a constant mean confining stress of 101kPa. Internal friction angles are shown in Figure 3 at the entire range of $\alpha$ varying with increments of $22.5^\circ$ from $0^\circ$ to $90^\circ$ and $b$ varying with increments of 0.25 from 0.0 to 1.0 in the sector I. $\Delta$ and $\tilde{\phi}$ are determined based on the internal friction angles at $\alpha = 0^\circ$, $b = 0$ ( $\phi_{0,0} = 41.20^\circ$ ) and $\alpha = 90^\circ$, $b = 0$ ( $\phi_{90,0} = 33.20^\circ$ ). The chosen parameters and the predicted strength of SMP and ESMP are presented in Figure 3. For clear expression, the data is divided into two groups for plotting. As shown in Figure 3 (a) and (b), the isotropic failure criterion with $\phi_{0,0} = 41.20^\circ$ clearly overestimates the measured strength when $\alpha \geq 45^\circ$, while the strength of soil-skeleton $\phi$ well reflects the average value of anisotropic geomaterials. As may be seen from the $\varphi - b$ diagram, ESMP fits the data well in the range of $b$ valuing from 0 to 0.3 at $\alpha = 0^\circ$ and $22.5^\circ$. The anisotropic failure criterion also shows a good agreement with the data when $b$ is larger than 0.7 and $\alpha$ varies from $45^\circ$ to $90^\circ$. Outside of these two ranges, the accuracy of ESMP is not very good. This is probably because that SMP is not
suitable for the shape of $\phi - b$ relationship of this test (Lade 2006). However, comparing with the extracted test data, ESMP captures the overall trend of the failure strength under the combined effect of $a$ and $b$.

![Figure 3. Comparisons of torsion shear test data on fine Nevada sand (Lade et al. 2014) with predication by the isotropic SMP failure criterion and the ESMP failure criterion in the $\phi - b$ diagram at (a) $\alpha = 0^\circ$, $22.5^\circ$ and $45^\circ$ (b) $\alpha = 67.5^\circ$ and $90^\circ$.](image)

**CONCLUSIONS**

A general 3-D anisotropic failure criterion is proposed for anisotropic geomaterials. The equivalent stress, proposed by Oda (1993), is redefined in theory through the equilibrium equation of soil-skeleton. With the application of the equivalent stress, the anisotropic criterion, based on the SMP failure criterion, is able to characterize the anisotropy of materials, the magnitude of the external stress and the rotational angles of the principal stresses. For cross-isotropic materials, the two material parameters $\Delta$ and $\bar{\phi}$ of the new anisotropic model can be determined by a pair of true triaxial shear tests, triaxial tests or torsion shear tests. Comparing with the test data, the anisotropic failure criterion can capture the strength anisotropy of soils in torsion shear tests performed to study the combined effects of the coefficient of intermediate principal stress and the stress rotation.

The equivalent stress and the anisotropic failure criterion presented in this paper can be conveniently incorporated into an existing model, such as the Drucker-Prager yield criterion and the Lade’s failure criterion. For the model based on the energy theory, for example the Cam-Clay model, the applicability of the modelling procedure with equivalent stress has to be verified in the future. Moreover, only the initial fabric tensor is considered as the induced anisotropy is too complicated which might limit the applicability of this anisotropic failure criterion.

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