Pricing and Numerical Simulation of Earthquake Catastrophic Bonds under the Semi-Markov Process

Qi Liang*
School of Mathematical Sciences, University of Electronic Science and Technology of China, Sichuan, China

*Corresponding author. Email: 1920075209@qq.com

Abstract. In recent years, the frequent occurrence of natural disasters and the increasing losses have made the traditional insurance and reinsurance markets insufficient in underwriting capacity. An alternative method for covering these losses is to transfer part of the catastrophic losses to financial market by issuing catastrophe-linked bonds. In this paper, we proposed a contingent claim model for pricing catastrophe risk bonds in a stochastic interest rate environment with the aggregate claims following a two-dimensional semi-Markov process where claims sizes follow heavy-tailed GPD distribution. Then, we estimated the parameters of the pricing model using real earthquake data in China from 1996 to 2017. Finally, we used Monte Carlo simulations to obtain the bond prices, and analyzed the numerical results.

Keywords: Catastrophe risk bonds, Contingent claim model, Semi-Markov, Monte Carlo.

1. Introduction
In recent years, natural disasters have occurred frequently, and losses have increased. The traditional insurance and reinsurance market have not enough ability to cover large amounts of insurance claims in a short period of time, and the limited risk transfer models have been unable to effectively disperse catastrophe risks. In order to expand the insurance and reinsurance company’ underwriting capacity, a series of catastrophe risk derivatives have emerged, and catastrophe (CAT) bonds are among the most successful innovation tools. Fair pricing is especially important for the issuance of catastrophe bonds.

In this paper, we assume the occurrence of an earthquake is independent of the financial market behavior and price the CAT bond in a stochastic rate environment. This article is divided into three important parts. First, we introduce the model proposed by [9] in a Markov-dependent environment. This model demonstrates the dependency between the claim sizes and the claim interval time, which can be understood that it is usually easier to reoccur in the short term after a serious earthquake. Second, we structure a classical zero-coupon payoff function and give the analytical formula for CAT bonds. At last, we use the actual earthquake loss data from 1996 to 2017 in China to estimate the relevant parameters of the pricing formula and give the dynamic prices of earthquake bonds.
2. Modeling

2.1. Valuation theory

Since the earthquake bonds can’t be hedged by a portfolio of traditional stocks and bonds, we need to consider an incomplete market, which has no universal pricing theory. In an arbitrage-free, there exists a risk-neutral measure $Q$ which is equivalent with real world measure $P$. The distribution and independence of variables under the $P$ measure will not change under the $Q$ measure, and the financial market behaves independently of the occurrence of catastrophes, see [2] for details. So, we propose a contingent claim model to obtain the $T$-year bond price $V_t$ at any time $t$ in asset pricing theory which can be expressed as follows,

$$V_t = E^Q_t(D(t,T)P_T|\mathcal{F}_t)$$

(1)

where $E^Q_t$ denotes expectation under risk-neutral measure $Q$, given $\mathcal{F}_t$. $\mathcal{F}_t$ represents all information about financial market and earthquake risk. $D(t,T) = \exp(-\int_t^T r(s) ds)$ is a stochastic discount factor. $P_T$ denotes the payoff function of the contingent claim. These two important parts are introduced in the next two sections.

2.2. Interest rate process

A stochastic discount factor can be characterized by an interest rate process. CIR model derived by Cox, Ingersoll, and Ross (1985) [10] introduced the square-root into the diffusion coefficient. In this paper, we use the CIR model to simulate stochastic interest rates. The instantaneous interest rate dynamics under the risk-neutral measure $Q$ can be expressed as

$$dr_t = k(\theta - r_t)dt + \sigma \sqrt{r_t}dW_t,$$

(2)

where $0 > \theta$ is the long-term average of the interest rate, $\sigma > 0$ is the volatility parameter of the interest rate, $k$ is a positive constant. Rate process $r_t$ will remain in positive domain given the condition $2k\theta \geq \sigma$. $W_t$ is a standard Brownian motion.

Based on the CIR model, we are able to get a $T$-year pure-discount bond price at time $t$ with following formula:

$$B_{CIR}(t,T) = A(t,T)e^{-C(t,T)r_t} = E^Q_P(e^{-\int_t^T r_t ds}).$$

(3)

with

$$A(t,T) = \left[ \frac{(k+\lambda r)(T-t)}{2k+(k+\lambda r)(e^{(T-t)}h-1)} \right]^{\frac{2k\theta}{\sigma^2}},$$

$$C(t,T) = \left[ \frac{-2h}{2k+(k+\lambda r)(e^{(T-t)}h-1)} \right],$$

$$h = \sqrt{(k + \lambda r)^2 + 2\sigma^2},$$

where $\lambda r$ represents the market price which is a constant. For more information, please refer to [11, 12]

2.3. A Semi-markov structure

Before designing the payout function of the earthquake bond in Eq.1, we need to construct the claims process and get the probability distribution of the aggregate claim losses. In this paper, we follow the semi-Markov structure proposed by [9] to add dependence between losses and interval time of claims. Define claims types as $\{J_n, n \geq 1\}$, which take their values in $J = \{1, \cdots, m\}(m \in \mathbb{N}^+)$ and can be characterized as different environment states with different arrival intensity and losses distributions. Let $\{X_n, n \geq 1\}$ and $\{T_n, n \in \mathbb{N}^+\}$ denote the claim sizes and the interval time respectively of the $n$th claim and define $U_n = T_n - T_{n-1}(n \in \mathbb{N}^+)$ which represents the sojourn time of the state $J_{n-1}$. 
Suppose that \( X_0 = 0, 0 < T_1 < T_2 < \cdots < T_n < T_{n+1} < \cdots \) and \( T_0 = U_0 = 0 \), and the trivariate process \( \{ (J_n, U_n, X_n); n \geq 0 \} \) is a semi-Markovian process, which can be defined by matrix \( Q (= Q_{ij}), i, j \in J \),

\[
Q_{ij}(t,x) = P(J_n = j, U_n \leq t, X_n \leq x|J_0, U_0, X_0), k = 1,2,\cdots, n-1, J_{n-1} = i.
\]  

(4)

where successive claims \( \{ n \} \) is an irreducible homogeneous continuous time Markov chain with transition matrix \( P(= (p_{ij}), i,j \in J) \). An interpretation of this model is that the arrival time before the next catastrophic event depends partially on the losses of the previous earthquake.

Assume random variable \( J_n \) and two-dimensional random variable \( (U_n, X_n), n \geq 1 \) are conditionally independent, then we have

\[
G_{ij}(t,x) = P(U_n \leq t, X_n \leq x|J_0, \cdots, J_{n-1} = i, J_n = j) = \{ \{ t_{i(20)} | t_{i(20)} \} \}_{i,j} \text{ if } p_{ij} > 0,
\]

where \( \mathbb{I} \) is an indicator function. And the conditional function of \( U_n \) and \( X_n \) can be denoted as follows,

\[
G_{ij}(t,\infty) = P(U_n \leq t|J_0, \cdots, J_{n-1} = i, J_n = j),
\]

\[
G_{ij}(\infty, x) = P(X_n \leq x|J_0, \cdots, J_{n-1} = i, J_n = j).
\]

By integrating \( J_n \), we can get the following equations:

\[
H_i(t,x) = P(U_n \leq t, X_n \leq x|J_0, \cdots, J_{n-1} = i) = \sum_{j=1}^{n} p_{ij} G_{ij}(t,x),
\]

\[
H_i(t,\infty) = P(U_n \leq t|J_0, \cdots, J_{n-1} = i),
\]

\[
H_i(\infty, x) = P(X_n \leq x|J_0, \cdots, J_{n-1} = i).
\]

Assume \( U_n \) and \( X_n \) are conditionally independent on \( J_n \), Eq 4 can be rewritten as:

\[
Q_{ij}(t,x) = p_{ij} G_{ij}(t,\infty) G_{ij}(\infty,x) \quad \forall t, x \in \mathbb{R}; \forall i,j \in J
\]

In order to get the distribution function of the accumulated claims, define \( L_n \) as the total claims amount till the \( n \)th claim, which can be expressed as:

\[
L_n = \sum_{k=1}^{n} X_k \quad \forall n \geq 1; \forall i,j \in J
\]

Thus, the joint probability of the trivariate process \( \{ (J_n, T_n, L_n); n \geq 0 \} \) is expressed as:

\[
P[J_n = j, T_n \leq t, L_n \leq x|J_0 = i] = Q_{ij}^n(t,x).
\]

The \( n \)-fold convolution of \( Q_{ij} \) can be valued recursively as:

\[
Q_{ij}^n(t,x) = \{ \{ 1-G_{ij}(t,\infty) \} \{ 1-G_{ij}(\infty,x) \} \} \quad \text{elsewhere},
\]

\[
Q_{ij}^n(t,x) = \sum_{k=1}^{n} \int_0^t \int_0^x Q_{ij}^{n-1}(t-t',x-x')dQ_{ij}(t',x')
\]

As before, integrate \( L_n \) and \( T_n \) we have:

\[
P[J_n = j, T_n \leq t|J_0 = i] = Q_{ij}^n(t,\infty) = \sum_{k=1}^{n} \int_0^t \int_0^x Q_{ij}^{n-1}(t-t',x-x')dQ_{ij}(t',x'),
\]

\[
P[J_n = j, L_n \leq x|J_0 = i] = Q_{ij}^n(\infty,x) = \sum_{k=1}^{n} \int_0^t \int_0^x Q_{ij}^{n-1}(\infty,x-x')dQ_{ij}(\infty,x-x').
\]

In addition,

\[
P[J_n = j|J_0 = i] = p_{ij}^n = \sum_{l=1}^{m} p_{ij}^n(i,\infty,x)|p_{ij}^n > 0
\]

\[
P[L_n \leq x|J_0 = i, J_n = j] = G_{ij}^n(\infty,x) = \{ \mathbb{I}_{\{t_{i(20)} \}} \}_{i,j} \text{ if } p_{ij}^n > 0.
\]

Then, one can obtain the following equation:

\[
Q_{ij}^n(t,x) = Q_{ij}^n(t,\infty) G_{ij}^n(\infty,x).
\]
Suppose the Markov Chain \( \{J_n, n \geq 1\} \) is ergodic and its stationary probability distribution \( \Pi_1, \Pi_2, \ldots, \Pi_m \in [0,1] \) exists, one can obtain the density function of the aggregate losses in [9].

Let \( F(t, x) \) denote the probability function of aggregate claims \( L(t) \) which are less than or equal to the threshold \( x \), at time \( t \). Then,

\[
F(t, x) = \sum_{i=1}^{m} \sum_{j=1}^{m} \Pi_i \sum_{n=0}^{\infty} \int_0^1 \left( 1 - H_i(t-t', \infty) \right) d \left[ Q_{ij}^n(t', \infty) G_{ij}^n(\infty, x) \right] .
\]

(5)

Assume the distribution of claim arrival time follow a Poisson process where the parameter depends only on the preceding claim and the parameters of claim amount process depends only on the next claim. Thus, we have:

\[
G_{ij}(t, \infty) = G_i(t, \infty) = \begin{cases} 
0 & t < 0 \\
1 - e^{\lambda_i t} & t \geq 0 
\end{cases},
\]

\[
G_{ij}(\infty, x) = G_i(\infty, x) = F_j(x).
\]

Then, we can get:

\[
Q_{ij}^n(t, \infty) = \left( p_{ij}(1 - e^{\lambda_i t}) \right)^n
\]

\[
G_{ij}^n(\infty, x) = \frac{\left( p_{ij} F_j(x) \right)^n}{p_{ij}^n}
\]

\[
H_i(t, \infty) = \sum_{l=1}^{m} p_{il}(1 - e^{\lambda_l t}) = 1 - e^{\lambda_i t}
\]

2.4. Pricing model
Consider a classical zero-coupon earthquake bond with following payoff functions at maturity time \( T \):

\[
P_{CAT}(T) = \begin{cases} 
\mu Z & L(T) \leq D \\
Z & L(T) > D
\end{cases}.
\]

(6)

where \( L(T) \) is the aggregate claims amount at maturity \( T \), \( D \) is the threshold value, and \( \mu \in [0,1] \) is the proportion of principle loss when aggregate loss is greater than the threshold.

Since the financial risks is independent of the earthquake risks, we can obtain the following pricing formula at time \( t \) with bond principle \( Z \) and maturity \( T \) by Eq. 2 as:

\[
V(t) = E^Q \left( e^{\int_t^T r(s) ds} P_{CAT}(T) | F_t \right) = E^Q \left( e^{\int_t^T r(s) ds} | F_t \right) E^Q \left( P_{CAT}(T) | F_t \right) = B_{CIR}(t, T) \left( Z P(L(T) \leq D) + \mu Z P(L(T) > D) \right)
\]

\[
= B_{CIR}(t, T) \left( Z [F(T, D) + (1 - \mu) F(T, D)] \right)
\]

\[
= B_{CIR}(t, T) \left( \mu + (1 - \mu) F(T, D) \right).
\]

3. Numerical simulation and analysis
Since it is extremely difficult to calculate high-order convolutions, we use Monte Carlo simulations via R software to obtain the numerical solution of pricing formula.

3.1. Data
The data set we use for simulation contains earthquake losses data for all records from China's 1996-2017 "Review of Earthquake Loss in China", including 249 non-zero data. In order to eliminate the impact of inflation, we used the 1996 consumer price index as the basis index to adjust the losses from 1997 to 2017.

3.2. Numerical calculation
Calculate the earthquake bond price with the face value \( Z = ¥1 \) at time \( t=0 \).
Step 1: We employ the 1-year maturity Chinese treasury bond rates (2002.01-2019.09) to fit the parameters of our CIR model by MLE method and assume the market risk price is -0.01. We obtain that the long-term mean interest rate $\theta$ is 0.0252 annually, the coefficient $k$ is 0.4715 and the volatility parameter $\sigma$ is 0.046.

Step 2: We define many claims period which more than two earthquake occurs in a month as state 1 ($J=1$), and the rest few claims period as state 2 ($J=2$). By analyzing our data set, we can conclude that there are 154 claims zoned as state 1, and 95 claims zoned as state 2. The transition probabilities and stationary distribution are showed in Table 1.

### Table 1. Transition probabilities.

<table>
<thead>
<tr>
<th></th>
<th>$j = 1$</th>
<th>$j = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{1j}$ for $j \in {1, 2}$</td>
<td>0.3789474</td>
<td>0.6210526</td>
</tr>
<tr>
<td>$P_{2j}$ for $j \in {1, 2}$</td>
<td>0.3856209</td>
<td>0.6143791</td>
</tr>
<tr>
<td>Stationary distribution</td>
<td>0.3830645</td>
<td>0.6169355</td>
</tr>
</tbody>
</table>

Step 3: In table 2, the earthquake economic losses have high skewness and kurtosis, so we can consider fitting the GPD distribution. In addition, in practice, half of the values that affect the pricing of earthquake bonds are those with large losses, namely the thick-tailed values. Therefore, we use the GPD distribution in the extreme value distribution theory.

### Table 2. Basic statistics of economic losses from earthquakes (unit: million yuan).

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>Max</th>
<th>Median</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.53</td>
<td>70154.72</td>
<td>67.05</td>
</tr>
<tr>
<td>Mean</td>
<td></td>
<td>Skewness</td>
<td>Kurtosis</td>
</tr>
<tr>
<td>1034.84</td>
<td>10.34354</td>
<td>124.7259</td>
<td></td>
</tr>
</tbody>
</table>

The threshold of GPD distribution of earthquake losses data in two states was determined by the hill graph method, and the model parameters were fitted based on the maximum likelihood method, as shown in Table 3.

### Table 3. Parameters of GPD distribution in two states.

<table>
<thead>
<tr>
<th></th>
<th>Threshold</th>
<th>Shape parameter</th>
<th>Scale parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>State1</td>
<td>40.75982</td>
<td>1.299957</td>
<td>40.75982</td>
</tr>
<tr>
<td>State2</td>
<td>33.50492</td>
<td>1.637824</td>
<td>90.103544</td>
</tr>
</tbody>
</table>

Fig. 1. and Fig. 2. demonstrate GPD distribution fits well in both two states.

![Figure 1. GPD fitting in state 1.](image1.png)
Step 4: We simply assume payoff function as \( \mu = 0.5 \) when the aggregate losses exceed the threshold \( D \). We now can illustrate the bond prices of earthquake at \( t=0 \) with the threshold level \( D \in [2928.11, 11712.46] \) and maturity time \( T \in [0.5, 2] \). Fig. 3 shows the bonds dynamic prices where the losses distribution follows the GPD distribution under the stochastic interest rate environment. From the process of the earthquake bonds prices change, we can see that the maturity of the earthquake bond has a more significant impact on the prices than the claim threshold.

4. Conclusion
In this study, we use a contingent claim process to price earthquake bond using models under a risk-free spot rate in a no-arbitrage market. Under the risk-neutral pricing measure, we propose a classic zero-coupon payoff function with claim trigger determined by the aggregate losses process following a semi-Markov dependent structure. Assume the spot interest rate follow CIR model and the inter-arrival time follow a homogeneous Poisson distribution.
The numerical analyses utilizing Monte Carlo simulations with real earthquake losses in China during 1996-2017 show that the bond price decreases as the threshold level decreases, as the bond maturity time increases. The results coincide with the reality situation. Because the larger the threshold is, the less likely the claim is to trigger, the lower the price of the bond should be. The longer the maturity time of the bond is, the greater the probability of catastrophe will be, the greater risk investors will take, and the lower the price of the bond should be to attract investors.

However, there are still some shortcomings in this paper. Although large data of earthquake losses are important, the impact of small losses on pricing may be ignored. There are many factors that affect the price of catastrophe bonds, such as moral hazard and basis risk in bond contracts. These factors are not fully considered in this paper, which is also an important research direction in the future.

References