Proportional Boundary Finite-element Method Applied to Two-dimensional Elastostatic Problem

Qiu-xiang LI\textsuperscript{1,2,*}, Ming-fu FU\textsuperscript{1}, Bang-hua XIE\textsuperscript{1}, Jiang-tao WEI\textsuperscript{1}, Zhi-hui LIAO\textsuperscript{1} and Zhi-wen ZOU\textsuperscript{1}

\textsuperscript{1}School of Civil Engineering and Architecture, Nanchang Institute of Technology, Nanchang, JiangXi, 330099, China
\textsuperscript{2}School of Civil Engineering and Architecture, Nanchang University, Nanchang, JiangXi, 330031, China
*Corresponding author

Keywords: Elastostatic problem, Virtual work principle, Scaled boundary finite-element.

Abstract. The control equation of the plane elastostatic problem under the scaled boundary coordinate system is derived by method of virtual work principle, and the solution of the governing equation is expounded. In a small example with a theoretical solution, we apply the proportional boundary finite element method to calculate and find that the accuracy of the result is very high.

Introduction

Proportional boundary finite-element method was first applied to continuum analysis as a novel semi-analytical approach by Wolf and Song\cite{1-2} and particularly suitable for fracture mechanics and unbounded domains\cite{3-5}. The governing equation from the virtual work principle does not require fundamental solution in the proportional boundary finite element method. We establish a coordinate system consisting of circumferential and radial directions. The circumferential boundary is discretized by finite elements and the governing equation is reduced to a radial ordinary partial differential equation\cite{6}. It is also suitable for elastic static problems\cite{7}. An example of applying proportional boundary finite element method to two-dimensional bounded plane elastic static problems is given in the paper.

Two-dimensional Elastic Mechanics Governing Equations in Scaled Coordinate System

Governing Equations

Equilibrium equation for two-dimensional elastic problem as following:

\begin{equation}
[L]\sigma + f = 0
\end{equation}

A proportional boundary coordinate system introduces by a scaling centre \((x_0, y_0)\) selected within the domain (figure 1) in proportional boundary finite-element method.

Figure 1. The scaling centre selected within the domain.

\(\xi\) runs from the scaling centre towards the boundary is the normalised radial coordinate, so the value at the center of the scale is 0, and the boundary is 1. Circumferential coordinate \(s\) specifies any value distance from the original boundary. So Cartesian coordinate \((x, y)\) is expressed by coordinates of the scaling center as follow:
\[
x = x_0 + \xi x_{(s)}, \quad y = y_0 + \xi y_{(s)}
\]  
(2)

\[
[L] = b^i \frac{\partial}{\partial \xi} + \frac{1}{\xi} b^i \frac{\partial}{\partial s} = \frac{1}{|J|} \begin{bmatrix} y_{(s)} - s & 0 \\ -x_{(s)} - s & y_{(s)} - s \end{bmatrix} \frac{\partial}{\partial \xi} + \begin{bmatrix} 0 & -x_{(s)} \\ -y_{(s)} & 0 \end{bmatrix}
\]

(3)

\[
|J| = x_{(s)} y_{(s)} - y_{(s)} x_{(s)} \]

For the calculation unit domain shown in figure 1, on the boundary with \( \xi = 1 \), the displacement of the discrete node is represented by a shape function similar to the node coordinates, and on any surface with a radial coordinate \( \xi \), displacement is introduced as

\[
u = u(\xi, s) = [N(s)]u(\xi)
\]

(4)

\[
\sigma(\xi, s) = D(b^i_{(s)} u_{(s)} + \frac{1}{\xi} b^i_{(s)} u_{(s)}) = D(B^i (u(\xi)))_{(s)} + \frac{1}{\xi} B^i u(\xi)
\]

(5)

\[
B^i = b^i_{(s)} [N(s)]_{(s)}, B^i = b^i_{(s)} ([N(s)])_{(s)}
\]

Substituting equation (4) and (5) into the principle of virtual work and taking the virtual displacement \( \delta u(\xi, s) = N(s) \delta u(\xi) \), the governing equation in scaled boundary coordinate system

\[
Q = E_0 u(\xi) + E_i u(\xi)
\]

(6)

\[
E_0 \xi^2 u(\xi)_{ss} + (E_0 + E_1 - E_i) \xi u(\xi)_{s} - E_i u(\xi) = 0
\]

(7)

In the formula

\[
E_0 = \int (B^i)^T \int_0^1 DB^i \int_0^1 [\delta u(\xi)] ds, E_1 = \int (B^i)^T \int_0^1 DB^i \int_0^1 [\delta u(\xi)] ds, E_i = \int (B^i)^T \int_0^1 DB^i \int_0^1 [\delta u(\xi)] ds
\]

Solution Procedure of the Governing Equation

Introduced a variable \( X(\xi) \), Equation (7) is transformed into a first-order ordinary differential equation with the number of unknowns

\[
X(\xi) = \begin{bmatrix} u(\xi) \\ Q(\xi) \end{bmatrix}
\]

(8)

\[
\xi X(\xi)_{s} = -Z X(\xi)
\]

(9)

With \( Q(\xi) \) being defined by

\[
Q(\xi) = E_0 \xi [u(\xi)]_{ss} + E_i u(\xi)
\]

As no parallel eigenvectors appear in feature decomposition, block diagonal Schur decomposition is performed on the Hamiltonian matrix

\[
Z = \begin{bmatrix} E_0 \xi E_i & -E_0 \\ -E_i + E_0 E_i E_i & -E_i E_0 \end{bmatrix}
\]

(10)

\[
Z = \begin{bmatrix} \psi^r & \psi^s \\ \psi^r & \psi^s \end{bmatrix} S = \begin{bmatrix} \psi^r & \psi^s \\ \psi^r & \psi^s \end{bmatrix}^{-1} = \psi S \psi^r
\]

(11)

Where the matrix \( S \) is a block-diagonal matrix in real Schur form and is sorted in ascending
order of its real parts eigenvalues. The real parts of the eigenvalues of the matrice \( S \) are non-positive and those of the matrice \( n \) are non-negative. \( S \) and \( n \) can be partitioned into 2N block matrices, respectively. The general solutions for \( u(\xi) \) and \( Q(\xi) \) from equation (8) as

\[
u(\xi) = \psi^0 e^{\xi/n} c_n + \psi^0 e^{\xi/n} c_n
\]

\[
Q(\xi) = \psi^0 e^{\xi/n} c_n + \psi^0 e^{\xi/n} c_n
\]

In a bounded domain \( (0 \leq \xi \leq 1) \), the displacements at \( \xi = 0 \) must remain finite. This condition \( \{u(\xi = 0)\} \) leads to \( \{c_n\} = 0 \) in equation (12a) and (12b) are reduced to

\[
u(\xi) = \psi^0 e^{\xi/n} c_n
\]

\[
Q(\xi) = \psi^0 e^{\xi/n} c_n
\]

The displacement on the boundary \( u(\xi = 1) \) determine the integration \( c_n \) as

\[
c_n = (\psi^0)^{-1} u(\xi = 1)
\]

The stiffness matrix \( K \) for a bounded medium can be determined by substituting equation (14) into equation (13b), so

\[
K = \psi^0 (\psi^0)^{-1}
\]

Combining equation (4) and (13a), displacement is expressed as

\[
u(\xi, s) = [N(s)\psi^0 e^{\xi/n} (\psi^0)^{-1} u(\xi = 1) = [N(s)\sum_{i=1}^{N} c_n e^{\xi/n} \phi_i]
\]

Substituting equation (16) into equation (5), the stress field can be obtained

\[
\sigma(\xi, s) = D(-B^T \psi^0 e^{\xi/n} (\psi^0)^{-1} u(\xi = 1) = D\sum_{i=1}^{N} c_n e^{\xi/n} [-\lambda_{B^T} + B^T] \phi_i]
\]

**Numerical Example**

As verifying the validity of the proportional boundary finite element method, the planar problem of bounded domain is calculated and compared with analytical solutions.

**Example 1-thick-walled Cylinder Subjected to Internal Pressure**

Taking a thick-walled cylinder subjected to uniform internal pressure as an example, the parameters of thick wall cylinder: inner diameter \( a=10cm \), outer diameter \( b= 20cm \), internal pressure is \( P=100Mpa \), elastic modulus of material is \( E=210GPa \), Poisson's ratio \( \nu = 0.3 \). According to the plane stress analysis, considering the symmetry, 1/4 of the structure can be taken as the analytical model. The analytical solution of the problem is \[8\]

\[
u = \frac{a^2 p(b^2 - a^2) + b^2 + r^2)}{Er(b^2 - a^2)}
\]

\[
\sigma(\xi) = \frac{-a^2 p(b^2 - a^2) + b^2 + r^2)}{r^2(b^2 - a^2)}
\]

As shown in figure 2.24 uniform nodes are used to discretize the boundary of the problem. The center of the analytical model is chosen as the scaling center.
Table 1 show the numerical solution and analytical solution for displacement and stress along the horizontal line at y=0.

Table 1. Displacement and stress of thick-walled cylinder subjected to internal pressure at y=0.

<table>
<thead>
<tr>
<th>number</th>
<th>coordinate</th>
<th>$u_x$ (mm)</th>
<th>$\sigma_z$ (Mpa)</th>
<th>$u_x$ (mm)</th>
<th>$\sigma_z$ (Mpa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50</td>
<td>0.049167</td>
<td>-97.19</td>
<td>0.049167</td>
<td>100.00</td>
</tr>
<tr>
<td>2</td>
<td>55</td>
<td>0.045810</td>
<td>-78.18</td>
<td>0.045811</td>
<td>76.86</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
<td>0.043112</td>
<td>-57.32</td>
<td>0.043111</td>
<td>59.26</td>
</tr>
<tr>
<td>4</td>
<td>65</td>
<td>0.040917</td>
<td>-46.24</td>
<td>0.040917</td>
<td>45.56</td>
</tr>
<tr>
<td>5</td>
<td>70</td>
<td>0.039119</td>
<td>-33.66</td>
<td>0.039119</td>
<td>34.69</td>
</tr>
<tr>
<td>6</td>
<td>75</td>
<td>0.037639</td>
<td>-26.31</td>
<td>0.037639</td>
<td>25.93</td>
</tr>
<tr>
<td>7</td>
<td>80</td>
<td>0.036417</td>
<td>-18.15</td>
<td>0.036417</td>
<td>18.75</td>
</tr>
<tr>
<td>8</td>
<td>85</td>
<td>0.035407</td>
<td>-13.03</td>
<td>0.035407</td>
<td>12.80</td>
</tr>
<tr>
<td>9</td>
<td>90</td>
<td>0.034574</td>
<td>-7.45</td>
<td>0.034574</td>
<td>7.82</td>
</tr>
<tr>
<td>10</td>
<td>95</td>
<td>0.033890</td>
<td>-3.75</td>
<td>0.033890</td>
<td>3.60</td>
</tr>
</tbody>
</table>

Conclusion

Governing equations of the proportional boundary finite element of the two-dimensional bounded is derived and solved in this paper. The method is applied to an example. The calculation results are agreement with the analytical solution friendly, which proves the validity and practicability of the method.

Acknowledgments

This work was financially supported by science and technology found project of Jiangxi Provincial Department (2013BBE50044, KJLD12048, GJJ-161120, GJJ180957), national college student innovation and entrepreneurship training program funding project(201611319007) and GanPo “555” talents Scheme project approved by Jiangxi provincial department of human resources.
References


