Upper Bounds for 4-rainbow Index of Graphs

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ABSTRACT

In this paper, we study 4-rainbow index \(\text{rx}_4(G)\) of \(G\). We first show that \(\text{rx}_4(G)\) is \(\text{rx}_4(G[D]) + 5\) for the connected graph \(G\) with minimum degree \(\delta(G) \geq 3\), where \(D\) is the connected 3-dominating set of \(G\). And then we determine a tight upper bound for \(K_{s,t}\) (\(4 \leq s \leq t\)) and a better bound for \((P_5,C_5)\)-free graphs. Finally, we obtain a sharp bound for 4-rainbow index of general graphs.

INTRODUCTION

All graphs considered in this paper are simple, connected and undirected. We follow the notations and terminology of [1] for those not defined here. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). For any vertex \(v\) of \(V(G)\), let \(N_G(v)\) denote the neighbor set of \(v\). And for any \(X \subseteq V(G)\), let \(G[X]\) and \(G\backslash X\) denote the subgraphs induced by \(X\) and deleting all vertices of \(X\) from \(G\), respectively. An edge-colored graph \(G\) is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number \(rc(G)\) of \(G\), introduced by Chartrand et al. [2], is the minimum number of colors that results in a rainbow connected graph \(G\). Later, another generalization of rainbow connection number was introduced by Chartrand et al.[3] in 2009. A tree \(T\) is a rainbow tree if no two edges of \(T\) have the same colors. Let \(k\) be a fixed integer with \(2 \leq k \leq n\). An edge coloring of \(G\) is called a \(k\)-rainbow coloring if for every set \(S\) of \(k\) vertices of \(G\), there exists a rainbow tree in \(G\) containing the vertices of \(S\). The \(k\)-rainbow index \(\text{rx}_k(G)\) of \(G\) is the minimum number of colors needed in a \(k\)-rainbow coloring of \(G\). It is obvious that \(rc(G) = \text{rx}_2(G)\). Let \(k\) be a positive integer. A subset \(D \subseteq V(G)\) is a \(k\)-dominating set of \(G\) if \(|N_G(v) \cap D| \geq k\) for every \(v \in V \backslash D\). The \(k\)-domination number \(r_k(G)\) is the minimum cardinality among the \(k\)-dominating sets of \(G\). Note that the 1-domination number \(r_1(G)\) is the usual domination number \(r(G)\). A subset \(S\)

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is a connected $k$-dominating set if it is a $k$-dominating set and the graph induced by $S$ is connected. The connected $k$-domination number $r''_k(G)$ is defined as the cardinality of a minimum connected $k$-dominating set. For $k = 1$, we write $r_c$ instead of $\gamma''_1(G)$.

Chandran et al. [4] used a strengthened connected dominating set (connected 2-way dominating set) to prove $r_c(G) \leq r_c(G[D]) + 3$. For 3-rainbow index, Li et al. obtained some basic results in [5,6]. In [6], Li et al. also investigated 4-rainbow index of a graph and obtained the following theorem.

**Theorem 1.1** [6] Let $G$ be a graph of order $n$. Then $r_4(G) = n-1$ if and only if $G$ is a tree, or an unicyclic graph, or a cactus with $c(G) = 2$, or $G \in G_1 \cup G_2$.

Here, let $G$ be a connected graph with $n$ vertices and $m$ edges, the cyclomatic number of $G$ as $c(G) = m-n+1$. A graph $G$ with $c(G) = k$ is called a $k$-cyclic graph. According to this, if a connected $G$ meets $c(G) = 0$, 1, 2, then $G$ is called acyclic (or a tree), unicyclic, bicyclic, respectively. $G_1$ is the set of graphs by identifying each vertex of $K_4$ with an end-vertex of an arbitrary path, and $G_2$ is the set of graphs by identifying each vertex of $K_4 \setminus e$ with the root of an arbitrary tree. A graph $G$ is a cactus if every edge is part of at most one cycle in $G$.

Recently, in [7], Liu and Hu did further investigation for 3-rainbow index of a graph and obtained some upper bounds of 3-rainbow index. In this paper, we focus our attention on 4-rainbow index.

**RESULTS OBTAINED FROM A CONNECTED 3-DOMINATING SET**

In this section, we give a sharp upper bound of 4-rainbow index of a graph in terms of a connected 3-dominating set.

**Theorem 2.1** Let $G$ be a connected graph with minimal degree $\delta \geq 4$. If $D$ is a connected 3-dominating set of $G$, then $r_4(G) \leq r_4(G[D]) + 5$ and the bound is sharp.

**Proof.** We prove the theorem by demonstrating that $G$ has a 4-rainbow coloring with $r_4(G[D]) + 5$ colors. For $x \in V(G \setminus D)$, its neighbors in $D$ will be called feet of $x$, and the corresponding edges will be called legs of $x$. We give $G[D]$ a 4-rainbow coloring using colors 1, 2, ..., $k(k = r_4(G[D]))$. Let $H = G \setminus D$. Partition $V(H)$ into sets $X, Y, Z$ as follows, where $Z$ is the set of all isolated vertices of $H$. In every nonsingleton connected component of $H$, choose a spanning tree. So we construct a forest on $W = V(H) \setminus Z$ and choose $X$ and $Y$ as any one of the bipartitions defined by this forest. We color every edge from $X$ to $D$ with $k + 1$ or $k + 2$ or $k + 3$, where each of $k + 1, k + 2, k + 3$ appears at least once and every edge from $Y$ to $D$ with $k + 1$ or $k + 2$ or $k + 4$, where each of $k + 1, k + 2, k + 4$ appears at least once and every edge between $X$ and $Y$ with $k + 5$. Since $G$ has the minimal degree $\delta \geq 4$, every vertex in $Z$ has at least four neighbors in $D$. We color three of them with $k + 1, k + 2, k + 4$ and all the others with $k + 5$. Next, we show that under such an edge coloring for any four vertices in $D$ there exists a rainbow tree containing them.

For four vertices $(x, y, z, w) \in D \times D \times D \times D$, there is already a rainbow tree containing them in $G[D]$. For four vertices $(x, y, z, w) \in D \times D \times D \times V(H)$, without of generality, we
suppose that \((x,y,z,w)\in D\times D\times D\times X\). We choose a leg of \(w\) arbitrarily, and let the corresponding foot be \(w'\). Since there is a rainbow tree containing \(x,y,z,w'\), there is a rainbow tree containing \(x,y,z,w\).

For four vertices \((x,y,z,w)\in D\times D\times V(H)\times V(H)\), we first consider that \((x,y,z,w)\in D\times D\times X\times X\). We choose the leg of \(z\) colored by \(k+1\) and the leg of \(w\) colored by \(k+2\), and let the corresponding feet be \(z'\) and \(w'\), respectively. Since there is a rainbow tree containing \(x,y,z,w'\), there is a rainbow tree containing \(x,y,z,w\). The conditions \((x,y,z,w)\in D\times D\times Y\times X\) and \((x,y,z,w)\in D\times D\times Z\times X\) are similar to the condition \((x,y,z,w)\in D\times D\times X\times X\). Next suppose \((x,y,z,w)\in D\times D\times X\times Y\). We choose the leg of \(z\) colored by \(k+1\) and the leg of \(w\) colored by \(k+2\), and let the corresponding feet be \(z'\) and \(w'\), respectively. Since there is a rainbow tree containing \(x,y,z,w'\), there is a rainbow tree containing \(x,y,z,w\). For the cases \((x,y,z,w)\in D\times D\times X\times X\) and \((x,y,z,w)\in D\times D\times Y\times X\), we can make similar discussions.

For four vertices \((x,y,z,w)\in D\times V(H)\times V(H)\times V(H)\), we first consider \((x,y,z,w)\in D\times X\times X\times X\). We choose the leg of \(z\) colored by \(k+1\) and the leg of \(z\) colored by \(k+2\) and the leg of \(w\) colored by \(k+3\), and let the corresponding feet be \(y'\) and \(z'\) and \(w'\), respectively. Since there is a rainbow tree containing \(x,y',z',w'\), there is a rainbow tree containing \(x,y,z,w\). The conditions \((x,y,z,w)\in D\times Y\times X\times X\) and \((x,y,z,w)\in D\times Z\times X\times X\) are the same as \((x,y,z,w)\in D\times X\times X\times X\). Next we consider \((x,y,z,w)\in D\times X\times X\times Y\). We choose the leg of \(y\) colored by \(k+1\) and the leg of \(z\) colored by \(k+2\) and the leg of \(w\) colored by \(k+4\), and let the corresponding feet be \(y'\) and \(z'\) and \(w'\), respectively. Since there is a rainbow tree containing \(x,y',z',w'\), there is a rainbow tree containing \(x,y,z,w\). For the cases \((x,y,z,w)\in D\times Y\times Y\times Z\), \((x,y,z,w)\in D\times Y\times Z\times Z\) and \((x,y,z,w)\in D\times X\times Y\times Y\), we can make similar discussions.

Then suppose \((x,y,z,w)\in D\times X\times Y\times X\). We choose the leg of \(y\) colored by \(k+1\) and the leg of \(z\) colored by \(k+4\) and the leg of \(w\) colored by \(k+5\), and let the corresponding feet be \(y'\) and \(z'\) and \(w'\), respectively. Since there is a rainbow tree containing \(x,y',z',w'\), there is a rainbow tree containing \(x,y,z,w\).

Finally suppose that \((x,y,z,w)\in V(H)V(H)\times V(H)\times V(H)\). We first consider \((x,y,z,w)\in X\times X\times Y\times Z\). We choose the leg of \(x\) colored by \(k+1\) and the leg of \(y\) colored by \(k+2\) and the leg of \(z\) colored by \(k+4\) and the leg of \(w\) colored by \(k+5\), and let the corresponding feet be \(x'\), \(y'\), \(z'\), \(w'\), respectively. Since there is a rainbow tree containing \(x',y',z',w'\), there is a rainbow tree containing \(x,y,z,w\). The conditions \((x,y,z,w)\in Y\times Y\times X\times Z\) and \((x,y,z,w)\in Z\times Z\times X\times Y\) are similar to the condition \((x,y,z,w)\in X\times X\times Y\times Z\). Next suppose that \((x,y,z,w)\in X\times X\times Y\times Y\). Since \(X\) and \(Y\) are one of the bipartition defined by the forest, any vertex in \(X\), there is a neighbor in \(Y\). Suppose \(x\) is adjacent to \(x''\) in \(Y\). We choose the leg of \(y\) colored by \(k+3\) and the leg of \(z\) colored by \(k+1\) and the leg of \(w\) colored by \(k+2\) and the leg of \(x''\) colored by \(k+4\), and let the corresponding feet be \(y',z',w',x'\), respectively. Since there is a rainbow tree containing \(x',y',z',w'\), there is a rainbow tree containing \(x,y,z,w\). For the cases \((x,y,z,w)\in X\times X\times Z\)
\((x,y,z,w) \in Y \times Y \times Z \times Z\), we can make similar discussion. For \((x,y,z,w) \in X \times X \times X \times Y\), we choose the leg of \(x\) colored by \(k+1\) and the leg of \(y\) colored by \(k+2\) and the leg of \(z\) colored by \(k+3\) and the leg of \(w\) colored by \(k+4\), and let the corresponding feet be \(x',y',z',w'\), respectively. Since there is a rainbow tree containing \(x',y',z',w'\), there is a rainbow tree containing \(x,y,z,w\). The conditions \((x,y,z,w) \in X \times X \times X \times Z\), \((x,y,z,w) \in Y \times Y \times Y \times Z\), \((x,y,z,w) \in Z \times Z \times Z \times X\), and \((x,y,z,w) \in Z \times Z \times Z \times Y\) are similar to the condition \((x,y,z,w) \in X \times X \times X \times Y\). Finally consider \((x,y,z,w) \in X \times X \times X \times X\). Suppose \(w\) is adjacent to \(x''\) in \(Y\). We choose the leg of \(x\) colored by \(k+1\) and the leg of \(y\) colored by \(k+2\) and the leg of \(z\) colored by \(k+3\) and the leg of \(x''\) colored by \(k+4\), and let the corresponding feet be \(x',y',z',w'\), respectively. Since there is a rainbow tree containing \(x',y',z',w'\), there is a rainbow tree containing \(x,y,z,w\). The condition \((x,y,z,w) \in Y \times Y \times Y \times Y\) is similar to the condition \((x,y,z,w) \in X \times X \times X \times X\).

**RESULTS FOR SOME SPECIAL GRAPHS**

In this section, we give upper bounds for 4-rainbow index of some special graphs.

**Theorem 3.1** For any complete bipartite graphs \(K_{s,t}\) with \(4 \leq s \leq t\), \(r_{4}(K_{s,t}) \leq 8\). Moreover, the bound is sharp.

**Proof.** Let \(U\) and \(W\) be the two partition sets of \(K_{s,t}\), where \(|U| = s\) and \(|W| = t\). Suppose \(U = \{u_1, u_2, \ldots, u_s\}\), \(W = \{w_1, w_2, \ldots, w_t\}\). Clearly we can find a connected 3-dominating set \(D = \{u_1, u_2, u_3, w_1, w_2, w_3\}\) of \(K_{s,t}\). In addition, \(K_{s,t} \setminus D\) is connected, by Theorem 2.1, \(r_{4}(K_{s,t}) \leq r_{4}(G[D]) + 5 = 8\). To prove the sharpness of the above upper bound, we derive the following claim.

**Claim.** For any \(s \geq 4\), \(t > 3 \times 8^s\), \(r_{4}(K_{s,t}) = 8\).

Firstly, we consider the graph \(K_{4,t}\). We may assume that there exists a 4-rainbow coloring \(c\) of \(K_{4,t}\) with \(k\) colors. Corresponding to this 4-rainbow coloring, for every vertex \(w\) in \(W\), there is a color code, \(\text{code}(w)\), assigned \(a_i = c(u_iw) \in 1, 2, \ldots, k\), \(1 \leq i \leq 4\). Observe that any four vertices in \(W\) have at least four distinct colors appeared in their color codes. Thus, we know that at most three vertices have the common code except possibly when \(a_1 \neq a_2 \neq a_3 \neq a_4\). Otherwise, there is no rainbow tree containing these four vertices which have the same code and at most three colors in color code. Therefore, when \(t > 3 \times k^3\), there must exist four vertices \(w', w'', w''', w''''\) such that \(\text{code}(w') = \text{code}(w'') = \text{code}(w''') = \text{code}(w''''\} = \{a_1, a_2, a_3, a_4\} \text{ and } a_1 \neq a_2 \neq a_3 \neq a_4\). If for a rainbow tree containing \(S = \{w', w'', w''', w''''\}\), it must contain \(u_1, u_2, u_3, u_4\) and \(w_i\) to guarantee its connectivity, where \(w_i\) belongs to \(W\) and \(\text{code}(w_i) = \{b_1, b_2, b_3, b_4\}\) and \(a_i, b_j\) are different from each other, \(i, j \in \{1, 2, 3, 4\}\). Thus \(k \geq 8\). So \(r_{4}(K_{s,t}) = 8\), when \(t > 3 \times 8^4\).
Similarly, we can prove \( r_{x_4}(K_s,t) = 8 \), for \( s \geq 5, t > 3 \times 8^s \). Thus, this claim also provides the tight proof of the Theorem 2.1.

For the graph \( K_{4,4} \) and \( K_{4,5} \), it is easy to prove that \( r_{x_4}(K_{4,4}) = 4 \) and \( r_{x_4}(K_{4,5}) = 5 \).

A graph \( G \) is called a perfect connected dominant graph if \( \gamma(X) = \gamma_c(X) \), for each connected induced subgraph \( X \) of \( G \). If \( G \) and \( H \) are two graphs, we say that \( G \) is \( H \)-free if \( H \) does not appear as an induced subgraph of \( G \). Furthermore, if \( G \) is \( H_1 \)-free and \( H_2 \)-free, we say that \( G \) is \( (H_1,H_2) \)-free. Next, we determine the upper bound for 4-rainbow index of \( (P_5,C_5) \)-free graphs.

**Lemma 3.2** [9] A graph \( G \) is a perfect connected-dominant graph if and only if \( G \) contains no induced path \( P_5 \) and induced cycle \( C_5 \).

**Lemma 3.3** [8] If \( G \) is a connected graph with \( \delta \geq 3 \), then \( \gamma(G) \leq \frac{3n}{8} \).

**Lemma 3.4** [7] Let \( G \) be a connected graph of order \( n \) with minimal degree \( \delta \geq 2 \). If \( D \) is a connected dominating set in a graph \( G \), then there is a set of vertices \( D' \supseteq D \) such that \( D' \) is a connected 2-dominating set and \( |D'| \leq \frac{1}{2} n + \frac{1}{2} |D| \).

**Lemma 3.5** Let \( G \) be a connected graph of order \( n \) with minimal degree \( \delta \geq 3 \). If \( D \) is a connected dominating set in a graph \( G \) and \( D' \) is a connected 2-dominating set in a graph \( G \), then there is a set of vertices \( D'' \supseteq D' \) such that \( D'' \) is a connected 3-dominating set and \( |D''| \leq \frac{3}{4} n + \frac{1}{4} |D| \).

**Proof.** There are two types of the components of \( G \setminus D' \): singletons and connected subgraphs. Let \( P \) be the set of the singletons, and \( Q \) be the set of the connected components of \( G \setminus D' \). Note that \( G \setminus D' = P \cup Q \). Since \( \delta \geq 3 \), for any vertex \( v \) in \( P \), it has at least three neighbors in \( D' \). In every non-singleton spanning forest on \( V(Q) \).

Choose \( X \) and \( Y \) as any one of the bipartition defined by this forest. Without loss of generality, we suppose that \( |X| \leq |Y| \).

Stage \( D'' = D' \)

while \( \exists v \in V(Q) \) such that \( |N(v) \setminus D'| = 2 \).

{if \( v \in Y \) Pick a vertex \( u \in N(v) \setminus X \)

Let \( D'' = D'' \cup \{u\} \) and \( Q = Q \setminus \{u\} \) else

\( D'' = D'' \cup \{v\} \) and \( Q = Q \setminus \{v\} \)}

Clearly \( D'' \) remains to be connected. Since stage ends only when any vertex in \( V(Q) \) has at least three neighbors in \( D'' \). So the final \( D' \) is a connected 3-dominating set. Let \( k \) be the number of iterations executed. Since we add a vertex in \( X \) to \( D'' \), \( |X| \) reduces by 1 in every iteration, \( k \leq |X| \leq \frac{1}{2} (n-|D'|) \), so \( |D''| \leq |D'| + k \leq |D'| + (n-|D'|) + \frac{1}{2} n + \frac{1}{2} |D'| \).

By Lemma 3.4, we have \( |D'| \leq \frac{1}{2} n + \frac{1}{2} |D| \). So
For a connected \((P_5,C_5)\)-free graph \(G\) with \(\delta \geq 4\), we can derive the following result by Theorem 2.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4.

**Theorem 3.6** For every connected \((P_5,C_5)\)-free graphs \(G\) with \(\delta \geq 4\), \(r_{x4}(G) \leq \frac{27}{32} n + 4\).

**Proof.** For every connected \((P_5,C_5)\)-free graphs \(G\) with \(\delta \geq 4\), from Lemma 3.2, \(\gamma(G) \geq \gamma c(G)\). And by Lemma 3.3, we have \(\gamma(G) \leq \frac{3}{8}n\). Thus, \(\gamma c(G) \leq \frac{3}{8}n\). Combining this with Lemma 3.4 and Lemma 3.5, the graph \(G\) has a connected 3-dominating set \(D\) and \(|D| \leq \frac{3}{4}n + \frac{1}{4}\). Observe that \(G[D]\) can get a 4-rainbow coloring using \(|D| - 1\) colors by ensuring that every edge of some spanning tree gets a distinct color. So by Theorem 2.1, we obtain \(r_{x4}(G) \leq r_{x4}(G[D]) + 5 \leq \frac{27}{32} n - 1 + 5 = \frac{27}{32} n + 4\).

**RESULTS FOR GENERAL GRAPHS**

In this section, we derive a sharp bound for 4-rainbow index of general graphs by block decomposition. And we also show a better bound for 4-rainbow index of general graphs with \(\delta(G) \geq 4\) by connected 3-dominating sets. Let \(A\) be the set of blocks of \(G\), whose element is \(K_2\); Let \(B\) be the set of blocks of \(G\), whose element is \(K_3\); Let \(C\) be the set of blocks of \(G\), whose element \(X\) is an unicyclic graph with \(|V(X)| \geq 4\) or a cactus with \(c(G) = 2\) and \(X \in G_1 \cup G_2\); Let \(D\) be the set of blocks of \(G\), whose element \(X\) is not an unicyclic graph with \(|V(X)| \geq 4\) and a cactus with \(c(G) = 2\) and \(X \in G_1 \cup G_2\).

**Theorem 4.1** Let \(G\) be a connected graph of order \(n(n \geq 4)\). If \(G\) has a block decomposition \(B_1,B_2,...,B_q\), then \(r_{x4}(G) \leq n - |D| - 1\). Moreover, the upper bound is sharp.

**Proof.** Let \(G\) be a connected graph of order \(n\) with \(q\) blocks in its block decomposition. If \(q = 1\), then we have done by Theorem 1.1 and \(r_{x4}(C_4) = 3\), which satisfies the above bound. Thus, we suppose \(q \geq 2\). Note that \(|A \cup B \cup C \cup D| = q\). From Theorem 1.1, we get \(r_{x4}(X) = |X| - 1\) for \(X \in C\) and \(r_{x4}(X) \leq |X| - 2\) for \(X \in D\). Hence, it follows that \(r_{x4}(G) \leq \sum_{X \in A} 1 + \sum_{X \in B} 2 + \sum_{X \in C} (|X| - 1) + \sum_{X \in D} (|X| - 2) = n - |D| - 1\).

In order to prove that the upper bound is sharp, we construct the graph \(G\) of order \(n\), as shown in Figure 1, consisting of \((n-4r-5)K_2, (2)K_3, (r)H\) where \(H\) is showed as Figure 1. It is clear that \(|A| = n-4r-5\), \(|B| = 2\), \(|D| = r\). We consider the size of a rainbow tree \(T\) containing the vertices \(u,v,w,y\). Since \(|E(T)| = 4 + 4r + n-5r-5 = n-r-1\), it follows that \(r_{x4}(G) \geq n-r-1\). By Theorem 4.1, we have \(r_{x4}(G) \leq n - |D| - 1 = n - r - 1\). So \(r_{x4}(G) = n - r - 1\).
We finish this section with general graphs with minimal degree at least 4. Here, we denote as $q_{\text{max}}(G)$ the maximum number of components of $G\setminus u$ among all vertices $u \in V$.

**Proposition 4.2** [10] Let $G$ be a connected graph on $n$ vertices with minimum degree $\delta \geq 2$ and let $k$ be an integer with $1 \leq k \leq \delta$. Then $\gamma^c_k \leq n - q_{\text{max}}(G)(\delta - k + 1)$.

For a general graph with $\delta \geq 4$, we obtain an upper bound for 4-rainbow index from Theorem 2.1 and Proposition 4.2.

**Proposition 4.3** Let $G$ be a connected graph with minimal degree $\delta \geq 4$. Then $r_{x_4}(G) \leq n - q_{\text{max}}(G)(\delta - 2) + 4$.

**Proof.** We obtain that $\gamma^c_3 \leq n - q_{\text{max}}(G)(\delta - 2)$ by Theorem 4.2. Then $r_{x_4}(G) \leq r_{x_4}(G[D]) + 5 \leq n - q_{\text{max}}(G)(\delta - 2) - 1 + 5 = n - q_{\text{max}}(G)(\delta - 2) + 4$ by Theorem 2.1.

**REFERENCES**

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