On Twin Edge Colorings of \( d \) Infinite Paths

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Abstract. Let \( \sigma \) be a proper edge coloring of a connected graph \( G \) of order at least 3, where the color set is \( \{0, 1, 2, \cdots, k-1\} \). If \( \sigma \) can induce a proper vertex coloring of \( G \), then \( \sigma \) is called a twin edge \( k \)-coloring of \( G \). The minimum number of colors for which \( G \) has a twin edge coloring is called the twin chromatic index of \( G \). In this paper, twin edge colorings of \( d \) infinite paths are studied, and it’s twin chromatic number is obtained.

Introduction

Let \( G=(V(G), E(G)) \) be a simple connected graph of order at least 3, where \( V(G) \) (or \( E(G) \)) is vertex (or edge) set of \( G \). And the degree of a vertex of \( G \) is the number of edges incident to \( v \), denoted by \( d_G(v) \). Meantime, the maximum degree of \( G \) is the maximum vertex degree in \( G \), denoted by \( \Delta(G) \). A proper edge coloring of \( G \) is a mapping \( f \) from \( E \) to \( N \) satisfying \( f(vw) \neq f(vz) \) for every pair of adjacent edges \( vw, vz \), for any vertex \( x \in V \), let \( S(x) \) denote the set of the colors of all edges incident to \( x \). The edge chromatic number of a simple graph \( G \) is the minimum number such that \( G \) has a proper edge-coloring, denoted \( \chi'(G) \).

Recently, an adjacent vertex distinguishing edge-coloring [1] of a simple graph \( G \) is a proper edge-coloring of \( G \) such that no pair of adjacent vertices has the same set of colors. The minimum number of colors of \( G \) is the adjacent vertex-distinguishing chromatic number, denoted by \( \chi'_a(G) \).

Few is known about adjacent vertex distinguishing edge-coloring. In 2006, Baril [2] show that the adjacent vertex-distinguishing chromatic number of the multidimensional mesh and the hypercube both are equal to the maximum degree of the both graphs plus one. In 2007, Balister [1] prove for bipartite graphs or such graphs with maximum degree \( \Delta(G)=3 \), it have \( \chi'_a(G) \leq \Delta(G)+2 \) or \( \chi'(G) \leq 5 \). In development, Wang [3] prove that if \( G \) is a \( k \)-minor free graph, without isolated edges, and with maximum degree \( \Delta(G) \geq 4 \) or \( \Delta(G) \geq 5 \), then \( \Delta(G) \leq \chi'_a(G) \leq \Delta(G)+1 \) or \( \chi'_a(G)=\Delta(G) \). In 2011, Frigerio [4] research that the adjacent vertex-distinguishing chromatic number of a regular graph, or a path, or a cycle. Meanwhile, Bu [5] show that \( \chi'_a(G) \) is at most the maximum degree plus 2 if \( G \) is a planar graph without isolated edges whose girth is at least 6. In 2012, Yan Chengchao [6] receive that the adjacent vertex-distinguishing chromatic number of a planar graph \( G \), with girth at least 4 and maximum degree at least 6, is less than maximal degree plus 2. For more results about the adjacent vertex distinguishing edge coloring, the readers may refer to [7-14].

More recently, the concept that is closely related to the adjacent vertex distinguishing edge-coloring of a graph \( G \) is a twin edge-coloring of a graph \( G \). The twin edge-coloring of a graph \( G \) is a proper edge coloring of \( G \) with the elements of the set \( \{0, 1, \cdots, k-1\} \) such that the induced vertex coloring in which the color of a vertex \( x \) in \( G \) is the sum of the colors of the edges incident with \( x \) is a proper vertex coloring. The twin chromatic number of \( G \) is the least number of colors set of \( G \), denoted by \( \chi'_t(G) \). In 2014, Andrews [15] put forward the concept of twin edge colorings of...
The vertex set of graph were studied. The product we are taking is the usual Cartesian product. Following Lemmas are established clearly.

**Proof** Theorem 1

For any connected graph $G$, we have $\chi'_2(G)\leq\chi'_1(G)$.

Here we address a natural extension of the very nice work by Jaradat [16] in which the chromatic number of products of graph were studied. The product we are taking is the usual Cartesian product. The vertex set of $G\times H$ is the Cartesian product $V(G)\times V(H)$ of the vertex sets of $G$ and $H$.

Let $P_\infty$ be a infinite path, and $V(P_\infty)=\mathbb{Z}$ (integer set) is vertex set of a infinite path $P_\infty$, where two vertices $i$ and $j$ are adjacent for every $i, j\in\mathbb{Z}$ if and only if $|i-j|=1$. And let $C_p(d)$ is the Cartesian product of $d$ infinite path, denoted by $C_p(d)=P_\infty\times P_\infty\times\cdots\times P_\infty$. Then the vertex set of $C_p(d)$ may be denoted $V(C_p(d))=\{(x_1, x_2, \cdots, x_d)\mid x_i\in\mathbb{Z}\}$, where $d\geq 2$.

In this paper, we investigate the twin edge-coloring of $C_p(d)$. Our goal here is to describe a somewhat general method to the twin edge-coloring, and give the chromatic number of $C_p(d)$. We refer to the books [17,18] for graph theory notation and terminology not defined in this paper.

**Main Results**

For the twin edge coloring of $C_p(d)$, we have the following result.

**Theorem 1** $\chi'_2(C_p(d))=2d+1$.

**Proof** Clearly, $\chi'_2(C_p(d))\geq\Delta(C_p(d))+1=2d+1$. To prove that $\chi'_2(C_p(d))\leq2d+1$, we construct the following edge coloring of $C_p(d)$: every edge $e_i=(x_1, \cdots, x_i, \cdots, x_d)(x_1, \cdots, x_i+1, \cdots, x_d)$ is assigned color $\sigma(e_i)=(A-2(i-1))\text{mod }p$, where $A=\sum_{i=1}^d x_i$, $p=2d+1$. Clearly, this coloring uses no more than $p$ colors.

Now, verifying $\sigma$ is a proper edge coloring of $C_p(d)$. Assume that $uv$ and $vw$ are any two adjacent edges of $C_p(d)$, where $v=(x_1, x_2, \cdots, x_i, \cdots, x_d)$, $u=(x_1, x_2, \cdots, x_i+\theta_1, \cdots, x_d)$, $w=(x_1, x_2, \cdots, x_i+\theta_2, \cdots, x_d)$, and $\theta_1, \theta_2=\pm 1$. To prove that $\sigma$ is a proper edge coloring of $C_p(d)$, it takes only verifying $\sigma(uv)\neq\sigma(vw)$. Obviously, when $i=k$, then $\theta_1=\theta_2=-1$. By the definition of the coloring $\sigma$, we have

$$\sigma(uv)=(A-2i+2+(\theta_1-1)/2)\text{mod }p,$$

$$\sigma(vw)=(A-2k+2+(\theta_2-1)/2)\text{mod }p.$$  

By definition of the coloring $\sigma$, $\sigma(uv)=\sigma(vw)$ is equivalent to $2(k-i)+(\theta_1-\theta_2)/2\equiv 0\text{mod }p$, which is impossible since $1\leq i, k\leq d < p/2$ and $1\leq 2(k-i)+(\theta_1-\theta_2)/2\leq 2d-1< p$. Thus $\sigma$ is a proper edge coloring of $C_p(d)$.

For any vertex $v=(x_1, x_2, \cdots, x_d)$, the vertex have $2d$ adjacent edges. By the definition of $\sigma$ and $\sigma'$, we have

$$S(v)=(2dA+4d)-\sum_{i=1}^d 4i,$$

$$\sigma'(v)=(2dA+4d-\sum_{i=1}^d 4i)\text{mod }p.$$  

Thanks to $v=(x_1, \cdots, x_i, \cdots, x_d)$, $\mu=(x_1, \cdots, x_i+\theta, \cdots, x_d)$ is any two adjacent vertex, where $\theta=\pm 1$, by (*), We have

$$\sigma'(\mu)=(2dA+\theta+4d-\sum_{i=1}^d 4i)\text{mod }p.$$  

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If $\sigma'(v) = \sigma'(\mu)$, then $\theta \equiv 0 \mod p$, therefore, $\sigma'(v) \neq \sigma'(\mu)$. It is clear that the coloring $\sigma'$ is a proper vertex colorings of $C_p(d)$.

We can see from the above analysis, $\sigma$ is $2d$-twin edge colorings of $C_p(d)$.

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References


