Linear Representation of Vector Group, Quick Solutions to Maximal Independent Group and Linear Equations in Linear Algebra

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Abstract. Through improving the process of "the linear representation of vector and vector group, and seeking solution to the maximal independent subset of vector group and linear equations" in linear algebra, this article simplifies the conventional two-step solution into one-step. The traditional two steps are: first, the matrix consisting of column vectors or the augmented matrix of linear equations is transformed into a ladder by the means of elementary row transformation; then this ladder matrix is transformed into a simplified row ladder matrix by doing one more elementary row transformation. However, with the improved method, we can work out solutions in one single step of elementary row transformation by transforming the transpose matrix of column vectors or the transpose augmented matrix of linear equations into a ladder matrix. The new method, thus, not only reduces half of the workload but also the difficulty of problem solving.

Introduction

In linear algebra, vector group, linear equations and matrix are closely related. Usually, a system of linear equations can be represented in three different ways, including the general form

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  &\vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

matrix form \(Ax = b\), and vector form \(\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n = b\). Similarly, a matrix can be represented by both row and column vector group. Further, the rank of matrix equals the rank of row vector group and column vector group.

An \(n\)-dimensional vector \(\beta\) can be expressed linearly by \(n\)-dimensional vector group \(\alpha_1, \alpha_2, \cdots, \alpha_n \iff \text{Non-homogeneous linear equation } \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n = \beta \text{ has solution.} \iff r(\alpha_1, \alpha_2, \cdots, \alpha_n) = r(\alpha_1, \alpha_2, \cdots, \alpha_n, \beta).

An \(n\)-dimensional vector group \(\alpha_1, \alpha_2, \cdots, \alpha_s\) is linear dependent. \(\iff \text{Homogeneous linear equation } \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n = 0 \text{ has non-zero solution.} \iff r(\alpha_1, \alpha_2, \cdots, \alpha_s) < s \iff \text{At least one vector in the vector set } \alpha_1, \alpha_2, \cdots, \alpha_s \text{ can be linearly represented by the other } s-1 \text{ vectors.}

An \(n\)-dimensional vector group \(\alpha_1, \alpha_2, \cdots, \alpha_s\) is linear independent. \(\iff \text{Homogeneous linear equation } \alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n = 0 \text{ only has zero solution.} \iff r(\alpha_1, \alpha_2, \cdots, \alpha_s) = s \iff \text{Each vector in the vector set } \alpha_1, \alpha_2, \cdots, \alpha_s \text{ cannot be linearly represented by the other } s-1 \text{ vectors.}

In the traditional textbook of linear algebra, the conventional method to solve "the linear representation of vector and vector group, the maximal independent subset of vector group and linear equations solution" is made up of two steps. In the first step, the matrix consisting of column
vectors or the augmented matrix of linear equations is transformed into a ladder by the means of elementary row transformation. In the second step, the ladder matrix is transformed into a simplified row ladder matrix by the same means. Such solution process involves multiple steps and the second step is more sophisticated than the first step, because column vectors must be transformed into the most simplified ladder shape.

Through continuous exploration and summary over many years' teaching practice, the author simplifies the two-step solution process into one-step process. The transpose matrix of column vectors or the transpose augmented matrix of linear equations is transformed into a ladder by the means of elementary row transformation. The new one-step solution process saves half workload and lowers the difficulty level of seeking solution. The simplification lies in the truth that when the row vectors are transformed into a ladder matrix, elementary transformation is only implemented on the rows.

Distinguish that a Vector Group is Linear Dependent or Independent

An $n$-dimensional vector group $\alpha_1, \alpha_2, \ldots, \alpha_s$ is linear independent. ⇔ Homogeneous linear equation $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_s x_s = 0$ only has zero solution. ⇔ $r(\alpha_1, \alpha_2, \ldots, \alpha_s) = s$ ⇔ Each vector in the vector set $\alpha_1, \alpha_2, \ldots, \alpha_s$ cannot be linearly represented by the other $s - 1$ vectors.

Example 1: Distinguish whether the vector group $\alpha_1 = (1, 0, 2, 5)^T$, $\alpha_2 = (0, 1, 3, 4)^T$, $\alpha_3 = (0, 0, 1, 4, 7)^T$, $\alpha_4 = (2, -3, 4, 11, 12)^T$ is linear dependent or not?

Traditional method:

$$
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 4 \\
2 & 3 & 4 & 11 \\
5 & 4 & 7 & 12
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 4 \\
0 & 0 & 7 & 14
\end{pmatrix}
,$$
linear independent

Improved method:

$$
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 4 \\
2 & -3 & 4 & 11
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & -14
\end{pmatrix}
,$$
linear independent

The Solution of the Rank and Maximal Independent Sunset of the Vector Group

In the traditional textbook of linear algebra, the conventional method to deal with "the linear representation of vector and vector group, the maximal independent subset of vector group and linear equations solution" involves two steps. In the first step, the matrix consisting of column vectors or the augmented matrix of linear equations is transformed into a ladder by the means of elementary row transformation. In the second step, the ladder matrix is transformed into a simplified row ladder matrix by another elementary row transformation.

Example 2. Work out the rank and maximal independent subset of the vector group

$$
\alpha_1 = (2, 1, 3, -1)^T, \quad \alpha_2 = (3, -1, 2, 0)^T, \quad \alpha_3 = (1, 3, 4, -2)^T, \quad \alpha_4 = (4, -3, 1, 1)^T
$$

And make the other vectors represented by the maximal independent subset.

Traditional method: $\Lambda = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. First step, transform $\Lambda$ into a ladder matrix by the means of elementary row transformation; Second, transform the ladder matrix into a simplified row ladder matrix by elementary row transformation.
\[
\begin{align*}
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
2 & 3 & 1 & 4 \\
1 & -1 & 3 & -3 \\
3 & 2 & 4 & 1 \\
-1 & 0 & -2 & 1 \\
\end{bmatrix}
&\rightarrow
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
1 & -1 & 3 & -3 \\
2 & 3 & 1 & 4 \\
3 & 2 & 4 & 1 \\
-1 & 0 & -2 & 1 \\
\end{bmatrix}
&\rightarrow
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
1 & -1 & 3 & -3 \\
0 & 5 & -5 & 10 \\
0 & 5 & -5 & 10 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
&\rightarrow
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
1 & -1 & 3 & -3 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
&\rightarrow
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
1 & 0 & 2 & -1 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\]

Therefore, \( r(\mathbf{A})=r(\alpha_1, \alpha_3, \alpha_4, \alpha_5)=2 \), maximal independent subset is \( \{\alpha_1, \alpha_5\} \), and \( \alpha_4 = -\alpha_1 + 2\alpha_2 \), \( \alpha_5 = 2\alpha_1 - \alpha_2 \).

**Improved method:** The only step is to transform the transpose matrix of column vectors or the transpose augmented matrix of linear equations into a ladder by the means of elementary row transformation.

\[
\begin{align*}
\alpha^T_1 (2 & 1 3 -1) \rightarrow (2 & 1 3 -1) \alpha^T_1 & \rightarrow r(\alpha_1, \alpha_3, \alpha_4) = r(A) = r(A^T) = 2 \\
\alpha^T_2 (3 & -1 2 0) \rightarrow (3 & -1 2 0) \alpha^T_2 & \rightarrow (\alpha_3 - 2\alpha_1 = -\alpha_2) \alpha^T_3 & \rightarrow (\alpha_3 = 2\alpha_1 - \alpha_2) \alpha^T_3 & \rightarrow (\alpha_4 + \alpha_1 = 2\alpha_2) \alpha^T_4 & \rightarrow (\alpha_4 = -\alpha_1 + 2\alpha_2) \alpha^T_4
\end{align*}
\]

maximal independent subset \( \{\alpha_1, \alpha_2\} \).

**Solution of Homogeneous Linear Equations**

Example 3. Solve the homogeneous linear equations:

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 - x_5 &= 0 \\
3x_1 + 2x_2 + x_3 + x_4 - 3x_5 &= 0 \\
x_2 + 2x_3 + 2x_4 + 6x_5 &= 0 \\
5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 &= 0
\end{align*}
\]

Traditional method:

\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 1 & 1 & -1 \\
3 & 2 & 1 & 1 & -3 \\
0 & 1 & 2 & 2 & 6 \\
5 & 4 & 3 & 3 & -1 \\
\end{bmatrix}
&\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 1 & -1 \\
0 & -1 & -2 & -2 & 0 \\
0 & 1 & 2 & 2 & 6 \\
0 & -1 & -2 & -2 & 4 \\
\end{bmatrix}
&\rightarrow
\begin{bmatrix}
1 & 0 & -1 & -1 & 0 \\
0 & -1 & -2 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
&\rightarrow
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
= c_1 \begin{bmatrix} 1 \\ -2 \\ 1+c_2 \\ 0 \\ 1 \\ \end{bmatrix}
\]

Improved method:

\[
\begin{align*}
\alpha^T_1 (1 & 3 0 5) \rightarrow (1 & 3 0 5) \alpha^T_1 & \rightarrow (\alpha_1 - \alpha_3 = 2\alpha_1 - 2\alpha_1) \alpha^T_1 & \rightarrow (\alpha_1 - \alpha_3 = -2\alpha_1 + 2\alpha_1) \alpha^T_1 & \rightarrow (\alpha_2 + \alpha_3 = 0) \alpha^T_2 & \rightarrow (\alpha_2 - \alpha_3 = 0) \alpha^T_2 & \rightarrow (\alpha_2 + \alpha_3 = 0) \alpha^T_2
\end{align*}
\]

The Linear Combination of Vector and Vector Group

An \( n \)-dimensional vector \( \mathbf{\beta} \) can be represented linearly by \( n \)-dimensional vector group \( \alpha_1, \alpha_2, \cdots, \alpha_n \) \( \leftrightarrow \) Non-homogeneous linear equation \( \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \beta \) has solution. \( \leftrightarrow r(\alpha_1, \alpha_2, \cdots, \alpha_n) = r(\alpha_1, \alpha_2, \cdots, \alpha_n, \beta) \).

**Example 4.** Distinguish the relationship between \( \mathbf{\beta} = (4,7,9,8)^T \) and the vector group \( \alpha_1 = (1,2,4,2)^T, \alpha_2 = (2,3,3,5)^T, \alpha_3 = (-3,-5,-9,-8)^T \).
Traditional method:

\[
\mathbf{A} = \begin{bmatrix}
1 & 2 & -3 & 4 \\
2 & 3 & -5 & 7 \\
4 & 3 & -9 & 9 \\
2 & 5 & -8 & 8
\end{bmatrix}
\]

Improved method:

\[
\begin{align*}
\mathbf{A}^* &= \begin{bmatrix}
1 & 2 & 4 & 2 \\
2 & 3 & 3 & 5 \\
-3 & -5 & -9 & -8 \\
4 & 7 & 9 & 8
\end{bmatrix} \\
\beta &= 3\alpha_1 + 2\alpha_2 + \alpha_3
\end{align*}
\]

\[
\begin{align*}
\alpha^* &= \begin{bmatrix}
1 & 2 & 4 & 2 \\
2 & 3 & 3 & 5 \\
-3 & -5 & -9 & -8 \\
4 & 7 & 9 & 8
\end{bmatrix} \\
\beta^* &= -4\alpha_1 + \alpha_2 + 3\alpha_3
\end{align*}
\]

**Solution of Non-homogeneous Linear Equations**

**Example 5.** Solve the nonhomogeneous linear equations

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 + x_5 &= 7 \\
3x_1 + 2x_2 + x_3 + x_4 - 3x_5 &= -2 \\
x_2 + 2x_3 + 2x_4 + 6x_5 &= 23 \\
5x_4 + 4x_2 - 3x_3 + 3x_4 - x_5 &= 12
\end{align*}
\]

**The two methods achieve the same purpose.**
Solution of Eigenvectors

Example 6. Solve the eigenvalues and eigenvectors of matrix \( A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \).

Solution:

\[
\lambda E - A = \begin{vmatrix} 
\lambda -1 & 1 & 1 & 1 \\
1 & \lambda -1 & 1 & 1 \\
1 & 1 & \lambda -1 & 1 \\
1 & 1 & 1 & \lambda -1 \\
\end{vmatrix} = (\lambda + 2) \begin{vmatrix} 1 & 1 & 1 & 1 \\
1 & \lambda -1 & 1 & 1 \\
1 & 1 & \lambda -1 & 1 \\
1 & 1 & 1 & \lambda -1 \\
\end{vmatrix} = (\lambda + 2)(\lambda - 2)^3 = 0
\]

\( \lambda_1 = -2 \), \( -2E - A = \begin{pmatrix} -3 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 \\
1 & 1 & -3 & 1 \\
1 & 1 & 1 & -3 \\
\end{pmatrix} , \begin{pmatrix} -3 \\
1 \\
1 \\
1 \\
\end{pmatrix} + \begin{pmatrix} 1 \\
-3 \\
1 \\
-3 \\
\end{pmatrix} = 0 \), \( \alpha_1 = \begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
\end{pmatrix} \)

\( \lambda_{2,3,4} = 2 \), \( 2E - A = \begin{pmatrix} 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix} , \begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
\end{pmatrix} = \begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
\end{pmatrix} \), \( \alpha_2 = \begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
\end{pmatrix} \), \( \alpha_3 = \begin{pmatrix} 0 \\
0 \\
0 \\
0 \\
\end{pmatrix} \), \( \alpha_4 = \begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
\end{pmatrix} \)

Example 7. Solve the eigenvalues and eigenvectors of real symmetric matrix \( A = \begin{pmatrix} 1 & 0 & 1 \\
0 & 2 & 0 \\
1 & 0 & 1 \end{pmatrix} \).

Solution:

\[
\lambda E - A = \begin{vmatrix} \lambda -1 & 0 & -1 \\
0 & \lambda -2 & 0 \\
-1 & 0 & \lambda -1 \end{vmatrix} = (\lambda - 2) \begin{vmatrix} \lambda -1 & -1 \\
-1 & \lambda -1 \end{vmatrix} = \lambda(\lambda - 2)^2 = 0
\]

\( \lambda_1 = 0 \), \( 0E - A = \begin{pmatrix} -1 & 0 & -1 \\
0 & -2 & 0 \\
-1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0 \end{pmatrix} \), \( x_3 = 0 \), \( x_2 = 0 \), \( x_1 + x_3 = 0 \), \( \alpha = \begin{pmatrix} -1 \\
0 \\
1 \end{pmatrix} \)

\( \lambda_{2,3} = 2 \), \( 2E - A = \begin{pmatrix} 1 & 0 & -1 \\
0 & 0 & 0 \\
-1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} \), \( x_3 = x_1 \), \( x_2 = 0 \), \( x_1 + x_3 = 0 \), \( \alpha_2 = \begin{pmatrix} 1 \\
0 \\
1 \end{pmatrix} \), \( \alpha_3 = \begin{pmatrix} 0 \\
0 \\
1 \end{pmatrix} \)

Conclusion

In summary, the two-step solution process is simplified to the one-step process, in which the transpose matrix of column vectors or the transpose augmented matrix of linear equations are transformed into a ladder by the means of elementary row transformation. The improvement not only reduces half of the workload but also lowers the difficulty of seeking solutions.

References