Mathmatic Abstraction for Fluid-structure Interaction Analysis of Bioprosthetic Heart Valves with Immersed Boundary Method

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ABSTRACT

Mathmatic abstraction, exactly variational formulations, is considered a fundamental work in engineering analysis. Fluid-structure interaction (FSI) is a field with no analytical solution, need increasingly use of numerical solutions. Designers and analysts devoted to this issue had paved a large path toward utilization of immersed boundary (IB) method. The study begines with mechanical modeling of involving materials. Due to the distinct mechanical response of aortic valves and its surrounding blood, normal numerical technique cannot achieve an approximation for precise hemodynamics analysis. In this paper, we introduced physically exact models for both fluid and solid, which are incompressible viscous Newton fluid and hyperelastic nonlinear anisotropic solid. we described the valves in a Lagrangian expression and used Eulerian coordinate for bloodstream. Then we set up the boundary conditions on the interfaces of both domains. Further, we applied IB method to avoid difficulties relating the two kinds of models by giving a reasonable term of force to approximate data transformation across the boundary. Finally, we gain several PDEs that govern the whole FSI system with specific initials.

KEYWORDS

Mathmatic abstraction; Bioprosthetic heart valves; Fluid-structure interaction; Immersed boundary method

INTRODUCTION

Heart valves are subjected to life-long hemodynamic forces, which plays an important role in cardiac function. More and more people suffer from valvular heart diseases and there are roughly 300,000 heart valve replacement because of valves dysfunction all over the world yearly.
(Yacouband & Takkenberg 2005) Bioprosthetic heart valves (BHV), part of effort to repair or replace those diseased ones, are primary choice with advantages of no need for anticoagulation. But are not such durable as their predecessors-mechanical heart valves. Biomedical engineers are searching for improvement to extend the service life of BHVs. Previous studies had been done with BHVs using a variety of computing techniques especially immersed boundary method (IBM). (Peskin 1972, Borazjani 2013) However, it is fundamentally crucial to build a mathematic model faithfully approximate the real situation of blood and leaflets in numerical simulations. With characters of continuity, elastic, viscous and distinct response under loading, these two materials can hardly be described with common models. Depictions about mechanicals of the flowing blood and leaflets have been proposed recent years considering more aspects vital to continuous and stable performs. Despite the commonly used constitutivemodel, there are several other models, namely phenomenological model, transversely isotropic models, aligned fiber model, etc, for valves. (Weinberg & Kaazempur-Mofrad 2005) The former has a microscopic expression, while the last shows fibers’ deformation contributing to the whole tissue. Boff introduced hyperelastic formulation in IBM. (Boffi et al. 2008) And Brust had discussed viscoelastic behavior of blood passing the valves. (Brust et al. 2013)

MATHMATIC MODELING OF FSI SYSTEM

This part of work basically includes three kinds of equations. Lagrangian form of solid, fluid equation with Eulerian variables as well as boundary conditions associating fluid and solid domain.

Firstly, a definition of coordinates are given. In this paper we generally let \( x \in \Omega_{f,t} \) denote the material point happens to be at position \( x \), and \( X \in \Omega_{s,t} \) be the material point that occupied the position \( X \) in the reference configuration. Here \( \Omega_{f,t} \) and \( \Omega_{s,t} \) are the time-depend fluid and solid domain, respectively, and we have \( \Omega_{f,t} \cup \Omega_{s,t} = \Omega \) which denotes the whole domain. Typically we made the initial configuration be the reference configuration. That is to say \( X \in \Omega_{s,t} \mid \ t=0 \). With this definition, we have variable \( x \) in Eulerian or physical coordinate, and \( X \) in Lagrangian or material coordinate. Association between the two variables are achieved by a mapping,

\[
\begin{align*}
x &= a(X, t) \\
X &= a^{-1}(x, t)
\end{align*}
\]

Then we let \( u(x, t) \) be the velocity of the material point whose physical coordinate value is \( x \) at time \( t \). For variable in Lagrangian form we naturally give velocity through (1), namely, \( u(a(X, t), t) \) indicates velocity of material point \( X \).

Note that \( a(X, t) \) denotes the physical coordinate of the material point \( X \) at time \( t \) is \( x \), and \( a^{-1}(x, t) \) means material \( X \) occurs at physical position \( x \) at time \( t \). Indeed, it is a Lagrange-Euler mapping as showed in Figure 1.
Solid Modeling

Bioprosthetic heart valves share similar structure with native ones, both of which are undergoing large deformation induced by blood flowing during aortic cycle and inversely affecting motion of fluid. Much work had been done to determine the stress-strain relationship of this functional material. Multiscale analysis focused on constitutive modeling ranging from phenomenological model with no information of the structure to unit-cell model in consideration of fibers. With observation from the angle of sub structure, there are uniaxially aligned fibers. Uniaxially alignment make the leaflets show an anisotropic property. (Billiar & Sacks 2000) In addition, the fibers’ wave contribute to the nonlinear response under loading. A majority part of water constituent leads to incompressibility. It is also the reason for viscosity, while the other part of material causes elasticity. (Weiss et al. 1996) A combination of this two response is called viscoelasticity, which has been proved to convert to hyperelastic due to dissipation of viscosity. (Fung 1993, Humphrey et al. 1987) All these properties are beyond the scope of elastic constitutive law. Holzapfel and Ogden’s constitutive model does a relatively good job to grasp these features. In this paper, we depicted the kinetic and stress state of the leaflets using this model. (Holzapfel & Ogden 2009)

Velocity and deformation tensor:

$$\frac{\partial \alpha(X, t)}{X} = u(\alpha(X, t), t)$$

(3)

$$F(X, t) = \frac{\partial \alpha(X, t)}{X}$$

(4)

With the constraint of impressibility we have

$$J = \det F(X, t) \equiv 1$$ during the whole process. Let $$B = FF^T, C = F^TF$$ be the left and right Cauchy-Green strain tensor, Green strain tensor be the Green strain tensor through an identity tensor I.

Then we introduce a three principal variants I1, I2, I3 of C and B, as well as two additional I4, I5 to depict anisotropic property.

$$I_1 = tr C$$

$$I_2 = \frac{1}{2} \left[ I_1^2 - tr(C^2) \right]$$

$$I_3 = |C|$$

(5)
With respect to the anisotropy, unit preferred vector $a$ can be useful to formulate variants.

$\mathbf{I}_a = a \cdot (C a)$

$\mathbf{I}_s = a \cdot (C^2 a)$  \hfill (6)

Strain-energy function and stress tensor:

Strain-energy functional $W(F)$ of the hyperelasticity, second Piola-Kirchhoff stress tensor $\sigma_p$ and corresponding Cauchy stress tensor $\sigma_s$ are given.

$\sigma_p = \frac{\partial W}{\partial F} F^{-T}$  \hfill (7)

$J\sigma_s = F \sigma_p F^{-T}$  \hfill (8)

$J\sigma_s = F \frac{\partial W}{\partial F}$  \hfill (9)

A Lagrangian multiplier $p$ is introduced to account for the incompressibility $J \equiv 1$, we have

$\sigma_s = F \frac{\partial W}{\partial F} - pI$  \hfill (10)

Due to the fact that variants I(i=1,2,3,4,5) could replace $F$ as independent variable of $W$, equ(7) could be rewritten as

$\sigma_s = F \sum_{i=1,i\neq 3}^{5} \frac{\partial W}{\partial F} - pI$  \hfill (11)

**Fluid Modeling**

Flows through the heart valves is generally considered to be in compressible Newtonian fluid. Navier-Stokes momentum equation and expression of fluid stress tensor are given.

$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \Delta \mathbf{u}$  \hfill (12)

$\sigma_f = -pI + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$  \hfill (13)

where $p(x, t)$ is the pressure of the fluid and $\mu$ is the viscosity. Note that $p$ here is to satisfy the incompressibility constraint.

**Modeling Boundary**

This part we naturally give the governing equations (respectively are(14), (15) and (16)) of no-slip, continuum of mass and stress.

$\mathbf{u}(x, t) = \mathbf{u}(\alpha^{-1}(X, t), t)$

$\nabla \cdot \mathbf{u} = 0$  \hfill (14)

$\sigma_f(x, t) \cdot n = \sigma_s(\alpha^{-1}(X, t), t) \cdot n$  \hfill (15)
\( n \) is the normal vector on the surface.

Initials:
\[ u(\mathbf{x}_0, t), p(\mathbf{x}_0, t) \] denote the inlet velocity and pressure of blood, actually quantities of left ventricular are be used as approximation.

\( \mathbf{a}(X, 0) \) denotes the position of valves’ first configuration.

**IMMERSED BOUNDARY METHOD**

**Philosophy of the Scheme**

Principle of virtual work is used to transform the continuum of stress in form of separate stresses to an integrate form by adding a volume force, which originates from the solid stress, to the fluid stress term.

A three dimensional Dirac delta \( \delta \) distribution is used to deal with difficulties associate two different forms of equations.

**Coupled Equations**

\[
\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f} \tag{17}
\]

\[
\nabla \cdot \mathbf{u} = 0 \tag{18}
\]

\[
\mathbf{f} = \int_{\Omega_{\text{inf}}} \nabla \cdot \mathbf{p} \delta(\mathbf{x} - \mathbf{a}) \, d\mathbf{X}
- \int_{\partial \Omega_{\text{inf}}} n \mathbf{p} \delta(\mathbf{x} - \mathbf{a}) \, dA(X) \tag{19}
\]

\[
\frac{\partial \mathbf{a}(X, t)}{\partial t} = \int_{\Omega} \mathbf{u}(x, t) \delta(x - \mathbf{a}(X, t), t) \, dx \tag{20}
\]

For the sake of simplicity, \( p, \alpha, n \) are in Lagrangian form, while \( f \) is in Eulerian form, \( \partial \Omega_{\text{ref}} \) is the boundary of reference configuration \( \Omega_{\text{ref}} \).

**CONCLUSION**

In this paper, we combine a new but relatively precise constitutive model of impressible, nonlinear, anisotropic and hyperelastic valve leaflets and the classical impressible Newton fluid model for blood passing through valves. This could ensure a more accurate expression of the physical behavior. The introduction of IBM creates an
easier condition to the FSI system solution. With this mathematic model depicting FSI of bioprosthetic heart valve, further work could be done to find numerical solutions that is better fitted the actually situation.

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