2016 International Conference on Applied Mathematics and Mechanics (ICAMM 2016)
ISBN: 978-1-60595-399-1

On the Vertex-Distinguishing Total Coloring of $P_n \vee P_n$ and $C_n \vee C_n$

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Keywords: Graph, Path, Cycle, Join-Graph, Vertex-Distinguishing total coloring.

Abstract. Let $G(V,E)$ be a simple graph, f is a mapping from $V(G) \cup E(G)$ to $\{1,2,\cdots,k\}$. Let $C_f(v) = \{f(v)\} \cup \{f(vw) | w \in V(G), vw \in E(G)\}$ for every $v \in E(G)$. If f is a K-proper-total-coloring, and for $\forall u,v \in V(G)$, we have $C_f(u) \neq C_f(v)$, then f is called the k-vertex-distinguishing total coloring (k-VDTC for shot). Let $\chi'_{vt}(G) = \min\{k | G$ has a k-vertex-distinguishing total colorint$\}$. Then $\chi'_{vt}(G)$ is called the vertex-distinguishing total chromatic number. The total chromatic number on $P_n \vee P_n$ and $C_n \vee C_n$.

Introduction

The coloring problem of graphs is widely applied in practice. In [1], some conditional coloring problems as introduced. Some network problem can be converted to the strong edge coloring [2−5] and adjacent strong edge coloring [6]. The graph considered in this paper are connected, finite, undirected and simple graph.

Definition 1[6] G is a simple graph and k is a positive integer, if it exists a mapping f, $V(G) \cup E(G) \xrightarrow{f} \{1,2,\cdots,k\}$, and satisfied with :

1) $\forall uv \in E(G)$, then $f(u) \neq f(v)$, $f(u) \neq f(uv) \neq f(v)$;
2) $\forall uv, uw \in E(G), v \neq w$, then $f(uv) \neq f(uw)$;
3) $\forall u, v \in V(G)$, then $C(u) \neq C(v)$

Then f is called a k-proper-total coloring of G. Let $C_f(u) = \{f(u)\} \cup \{f(uw) | w \in V(G), uw \in E(G)\}$, and $\overline{C}_f(u) = \{1,2,\cdots,k\} - C_f(u)$ for every $u \in E(G)$. If $\forall u,v \in V(G)$, we have $C_f(u) \neq C_f(v)$, and $\overline{C}_f(u) \neq \overline{C}_f(v)$, then f is called k-vertex-distinguishing total coloring. The number min{k|G has k-vertext-distinguishing total coloring} is called the vertex-distinguishing total chromatic number and denoted by $\chi'_{vt}(G)$.

Definition 2[6] Suppose G and H are two simple graphs which are vertex disjointed and edge disjointed,

$$V(G \vee H) = V(G) \cup V(H),$$
$$E(G \vee H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\},$$

Then $G \vee H$ is called a Join-graph of G and H.

Definition 3[7] For a connected graph G, n_i denotes the number of vertex which has degree i, δ, Δ denoted the minimum, maximum degree of G respectively, then is called total combinatorial degree of G.

$$\mu(G) = max\left\{k = min\left\{\lambda | \binom{\lambda}{d_i + 1} \geq n_i, \delta \leq d_i \leq \Delta\right\}\right\}$$

For vertex-distinguishing total chromatic number, we will give a conjecture as follow:

Conjecture 1[8] for every graph G with at least order 2, we have

$$\mu(G) \le \chi'_{vt}(G) \le \mu(G)+1,$$

In this paper, we obtain the vertex-distinguishing chromatic number of $P_n \vee P_n$ and $C_n \vee C_n$. For the graph-theoretic terminology the reader is referred to [9-12].

Lemma 1 For $n \ge 2$, then

$$\mu(P_n \vee P_n) = \begin{cases} 5, & n=2; \\ n+4, & n=3,4,5,6,7,8 \\ n+5, & n \ge 9. \end{cases}$$

Lemma 2

$$\mu(C_n \vee C_n) = \begin{cases} n+4, & n=3,4; \\ n+5, & \ge 5. \end{cases}$$

Vertex-Distinguishing Total Coloring of $P_n \vee P_n$

Theorem 1 Let P_n be a path with order n, then

$$\chi'_{vt}(P_n \vee P_n) = \begin{cases} 5, & n=2; \\ n+4, & 3 \le n \le 6; \\ n+5, & n \ge 7. \end{cases}$$

Proof Let two path be $u_1u_2, u_2u_3, \cdots, u_{n-1}u_n$ and $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$, there are three cases to be discussed as follow:

Case 1 When $n=2$, then $P_2 \vee P_2 = K_4$, the conclusion is true by [1].

Case 2 When $3 \le n \le 6$, we have four subcases to discussed as follow:

Subcase 2.1 When $n=3$, we constructed a map f from $V(P_3 \vee P_3) \cup E(P_3 \vee P_3)$ to $C=\{1,2,3,4,5,6,7\}$:

$$f(u_i)=i, i=1,2,3;\ f(u_iv_j)=i+j, i=1,2,3; j=1,2,3; f(u_1u_2)=6;\ f(u_2u_3)=7;$$
$$f(v_i)=j+4, j=1,2,3; f(v_1v_2)=1;\ f(v_2v_3)=2;$$

So we have

$$\overline{C}(u_1)=\{5,7\};\ \overline{C}(u_2)=\{1\};\ \overline{C}(u_3)=\{1,2\}\overline{C}(v_1)=\{6,7\};\ \overline{C}(v_2)=\{7\};\ \overline{C}(v_3)=\{1,3\}$$

Obviously, the f is a 7-VDTC of $P_3 \vee P_3$, the conclusion is true.

Subcase 2.2 When $n=4$, we constructed a map f from $V(P_4 \vee P_4) \cup E(P_4 \vee P_4)$ to $C=\{1,2,3,4,5,6,7,8\}$:

$$f(u_i)=i, i=1,2,3,4;$$
$$f(u_iv_j)=i+j(when\ i+j>7, take\ mod\ 7), i=1,2,3,4; j=1,2,3,4;$$
$$f(u_1u_2)=f(u_3u_4)=7;\ f(u_2u_3)=8;$$
$$f(v_1)=f(v_3)=8;\ f(v_2)=f(v_4)=7; f(v_1v_2)=1;\ f(v_2v_3)=2;\ f(v_3v_4)=3$$

So we have

$$\overline{C}(u_1)=\{6,8\};\ \overline{C}(u_2)=\{1\};\ \overline{C}(u_3)=\{2\}; \overline{C}(u_4)=\{3,8\}$$
$$\overline{C}(v_1)=\{6,7\};\ \overline{C}(v_2)=\{7,8\};\ \overline{C}(v_3)=\{7\}; \overline{C}(v_4)=\{4,8\}$$

Obviously, the f is a 8-VDTC of $P_4 \vee P_4$, the conclusion is true.

Subcase 2.3 When $n=5$, we constructed a map f from $V(P_5 \vee P_5) \cup E(P_5 \vee P_5)$ to $C=\{1,2,3,4,5,6,7,8,9\}$:

$$f(u_i)=i, i=1,2,\cdots,5;$$
$$f(u_iv_j)=i+j(when\ i+j+1>8, take\ mod\ 8), i=1,2,\cdots,5; j=1,2,\cdots,5;$$
$$f(u_1u_2)=f(u_3u_4)=8;\ f(u_2u_3)=f(u_4u_5)=9;$$
$$f(v_1)=f(v_3)=8;\ f(v_2)=f(v_4)=7; f(v_jv_{j+1})=j+1, j=1,2,3,4.$$

So we have

$$\overline{C}(u_1)=\{2,9\};\ \overline{C}(u_i)=\{i+1\}, i=2,3,4;\ \overline{C}(u_5)=\{1,6\}$$

$$\overline{C}(v_1) = \{1,9\};\ \overline{C}(v_2) = \{1\};\ \overline{C}(v_3) = \{9\}; \overline{C}(v_4) = \{8\}; \overline{C}(v_5) = \{6,9\}$$

Obviously, the f is a 9-VDTC of $P_5 \vee P_5$, the conclusion is true.

Subcase 2.4 When $n = 6$, we constructed a map f from $V(P_6 \vee P_6) \cup E(P_6 \vee P_6)$ to $C = \{1,2,3,4,5,6,7,8,9,10\}$:

$$f(u_1) = f(u_4) = 1; f(u_2) = f(u_5) = 2; f(u_3) = f(u_6) = 3$$
$$f(u_i v_j) = i + j + 2\big(when\ i + j + 2 > 10, f(u_i v_j) = i + j - 5\big), i = 1,2,\cdots,5; j = 1,2,\cdots,5;$$
$$f(u_6 v_1) = 9;\ f(u_6 v_2) = 2;\ f(u_6 v_j) = j + 2, j = 3,4,5,6;$$
$$f(u_1 u_2) = f(u_4 u_5) = f(v_1 v_2) = f(v_4 v_5) = 3;$$
$$f(u_2 u_3) = f(u_5 u_6) = f(v_2 v_3) = f(v_5 v_6) = 1;$$
$$f(u_3 u_4) = f(v_3 v_4) = 2;\ f(v_1) = 10;\ f(v_j) = j + 3, j = 2,3,4,5,6.$$

So we have

$$\overline{C}(u_1) = \{2,10\};\ \overline{C}(u_i) = \{i + 2\}, i = 2,3,4,5;\ \overline{C}(u_6) = \{8,10\}$$
$$\overline{C}(v_1) = \{1,2\};\ \overline{C}(v_2) = \{10\};\ \overline{C}(v_3) = \{3\}; \overline{C}(v_4) = \{1\}; \overline{C}(v_5) = \{2\};\ \overline{C}(v_6) = \{2,3\}$$

Obviously, the f is a 10-VDTC of $P_6 \vee P_6$, the conclusion is true.

Case 3 When $n \geq 7$, there are two subcases to be considered.

Subcase 3.1 When $n = 7$, we prove $\chi'_{vt}(P_7 \vee P_7) \geq 12$ firstly. Otherwise, we have $\chi'_{vt}(P_7 \vee P_7) = 11$ from lemma1.

Let f be 11-VDTC of $P_7 \vee P_7$, supposing $\overline{C}(w) = \{1,2,\cdots,11\} \backslash C(w)$, $w \in \{u_i | 1,2,\cdots,7\} \cup \{v_i | 1,2,\cdots,7\}$. Form the definition 1 and 2, for f we have

$$|\overline{C}(u_i)| = |\overline{C}(v_i)| = 1, i = 2,3,\cdots,6; |\overline{C}(u_i)| = |\overline{C}(v_i)| = 2, i = 1,7;$$

and

$$\begin{cases} \overline{C}(u_i) \in \overline{C}(u_1) = \overline{C}(u_7) = \overline{C}(v_1) = \overline{C}(v_7), i = 2,3,4,5,6; \\ \overline{C}(v_i) \in \overline{C}(u_1) = \overline{C}(u_7) = \overline{C}(v_1) = \overline{C}(v_7), i = 2,3,4,5,6; \end{cases} \quad (1)$$

But

$$\left| \cup_{i=2}^{6} \big(\overline{C}(u_i) \cup \overline{C}(v_i)\big) \right| = 10 > \left| \overline{C}(u_1) \cup \overline{C}(u_7) \cup \overline{C}(v_1) \cup \overline{C}(v_7) \right| \leq 8 \quad (2)$$

There is a contradiction by (1) and (2), so there is no 11-VDTC of $P_7 \vee P_7$. It is obviously to prove exists 12-VDTC of $P_7 \vee P_7$, we constructed a map f from $V(P_7 \vee P_7) \cup E(P_7 \vee P_7)$ to $C = \{1,2,\cdots,12\}$:

$$f(u_1) = f(u_4) = f(u_7) = 1; f(u_2) = f(u_5) = 2; f(u_3) = f(u_6) = 3$$
$$f(u_i v_j) = i + j + 2\big(when\ i + j + 2 > 12, f(u_i v_j) = i + j - 9\big), i = 1,2,\cdots,6; j = 1,2,\cdots,7;$$
$$f(u_7 v_1) = 11;\ f(u_7 v_2) = 12;\ f(u_7 v_j) = j + 1, j = 3,4,5,6,7;$$
$$f(u_2 u_3) = f(u_5 u_6) = f(v_1 v_2) = f(v_4 v_5) = 1;$$
$$f(u_3 u_4) = f(u_6 u_7) = f(v_2 v_3) = f(v_5 v_6) = 2;$$
$$f(u_1 u_2) = f(u_4 u_5) = f(v_3 v_4) = f(v_6 v_7) = 3; f(v_1) = 12;\ f(v_j) = j + 3, j = 2,3,\cdots,7.$$

So, we have

$$\overline{C}(u_1) = \{2,11,12\}; \overline{C}(u_2) = \{4,12\}\ \overline{C}(u_i) = \{i + 1, i + 2\}, i = 3,4,5,6;$$
$$\overline{C}(u_7) = \{3,9,10\} \overline{C}(v_1) = \{2,3,10\};\ \overline{C}(v_2) = \{3,11\};\ \overline{C}(v_3) = \{1,12\};$$
$$\overline{C}(v_j) = \{(j\ mod\ 3) + 1\ \}, j = 4,5,6;\ \ \overline{C}(v_7) = \{1,2,7\}$$

Obviously, the f is a 12-VDTC of $P_7 \vee P_7$, the conclusion is true.

Subcase 3.2 When $n \geq 8$, we constructed a map f from $V(P_n \vee P_n) \cup E(P_n \vee P_n)$ to $C = \{1,2,\cdots,n + 4,0\}$:

Coloring the edges $u_1u_2, u_2u_3, \cdots, u_{n-1}u_n$ with colors 3,1,2 repetitively, coloring the vertices of $u_1, u_2, \cdots, u_{n-1}, u_n$ with color 1,2,3 repetitively;

$$f(u_i v_j) = i + j + 2\big(when\ i + j + 2 > n + 5, f(u_i v_j) = i + j - n\big), i = 1,2,\cdots,n; j = 1,2,\cdots,n;$$

Coloring the vertices $v_1, v_2, \cdots, v_{n-1}, v_n$ with colors 1,2,3 repetitively;

$$f(v_1) = n + 4; f(v_2) = n + 5;\ f(v_j) = j + 2, j = 3,4,\cdots,n.$$

So, we have

$$\overline{C}(u_1) = \{2, n + 4, n + 5\}; \overline{C}(u_2) = \{4, n + 5\};\ \overline{C}(u_i) = \{i + 1, i + 2\}, i = 3,4,\cdots,n - 1;$$

When $n \equiv 1 \ mod\ 3$, $\overline{C}(u_n) = \{3, n+1, n+2\}$;
When $n \equiv 2 \ mod\ 3$, $\overline{C}(u_n) = \{1, n+1, n+2\}$;
When $n \equiv 0 \ mod\ 3$, $\overline{C}(u_n) = \{2, n+1, n+2\}$;

$$\overline{C}(v_1) = \{2,3, n+5\};\ \overline{C}(v_j) = \{(j \ mod\ 3) + 1, j+2\}, j = 2,3, \cdots, n-1;$$

When $n \equiv 1 \ mod\ 3$, $\overline{C}(v_n) = \{1,2, n+2\}$;
When $n \equiv 2 \ mod\ 3$, $\overline{C}(v_n) = \{2,3, n+2\}$;
When $n \equiv 0 \ mod\ 3$, $\overline{C}(v_n) = \{1,3, n+2\}$;
Obviously, the f is a (n+5)-VDTC of $P_n \vee P_n$, the conclusion is true.
From all of above, the conclusion is true.

Vertex-Distinguishing Total Coloring of $C_n \vee C_n$

Theorem 2 For $n \geq 3$, then

$$\chi'_{vt}(C_n \vee C_n) = \begin{cases} n+4, & n = 3; \\ n+5, & n \geq 4. \end{cases}$$

Proof Let two cycles are $V(C_n) = u_1u_2, u_2u_3, \cdots, u_{n-1}u_n$ and $V(C_n) = v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$ respectively, there are two cases to be discussed as follow:

Case 1 When $n = 3$, then $C_3 \vee C_3 = K_6$, the conclusion is true by [1].

Case 2 When $n \geq 4$, $\chi'_{vt}(C_n \vee C_n) \geq (n+5)$ according to lemma 3, we only need to prove $C_n \vee C_n$ exists (n+5)-VDTC, so we constructed a map f from $V(C_n \vee C_n) \cup E(C_n \vee C_n)$ to $C = \{1,2, \cdots, n+5\}$:

Subcase 2.1 When $n \equiv 0 \ mod\ 2$,

Coloring the edges $u_1u_2, u_2u_3, \cdots, u_{n-1}u_n, u_nu_1$ with colors n+4,n+5 repetitively, coloring the vertices of $u_1, u_2, \cdots, u_{n-1}, u_n$ with color 1,2 repetitively;

Coloring the edges $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n, v_nv_1$ with colors 1,2 repetitively, coloring the vertices of $v_1, v_2, \cdots, v_{n-1}, v_n$ with color n+4,n+5 repetitively;

$$f(u_iv_j) = i+j+1 (when\ i+j+1 > n+3, f(u_iv_j) = i+j-n), i = 1,2, \cdots, n-1; j = 1,2, \cdots, n;$$

$$f(u_nv_1) = n+3;\ f(u_nv_j) = j+1, j = 2,3, \cdots, n;$$

So we have

$$\overline{C}(u_1) = \{2, n+3\};$$

When $i \equiv 0 \ mod\ 2$, $\overline{C}(u_i) = \{1, i+1\}, i = 2,3, \cdots, n-1$;
When $i \equiv 1 \ mod\ 2$, $\overline{C}(u_i) = \{2, i+1\}, i = 2,3, \cdots, n-1$;

$$\overline{C}(u_n) = \{1, n+2\}; \overline{C}(v_1) = \{n+2, n+5\};\ \overline{C}(v_2) = \{n+3, n+4\};$$

When $i \equiv 0 \ mod\ 2$, $\overline{C}(v_i) = \{i, n+4\}, i = 3,4, \cdots, n-1$;
When $i \equiv 1 \ mod\ 2$, $\overline{C}(v_i) = \{i, n+1\}, i = 3,4, \cdots, n-1$;

$$\overline{C}(v_n) = \{n, n+4\}.$$

Subcase 2.2 When $n \equiv 1 \ mod\ 2$,

Coloring the edges $u_1u_2, u_2u_3, \cdots, u_{n-1}u_n$ with colors n+4,n+5 repetitively, $f(u_nu_1) = 2$, coloring the vertices of $u_1, u_2, \cdots, u_{n-1}, u_n$ with color 1,2 repetitively;

Coloring the edges $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$ with colors 1,2 repetitively, ,$f(v_nv_1) = n+5$, coloring the vertices of $v_1, v_2, \cdots, v_{n-1}, v_n$ with color n+4,n+5 repetitively;

$$f(u_iv_j) = i+j+1 (when\ i+j+1 > n+3, f(u_iv_j) = i+j-n), i = 1,2, \cdots, n-1; j = 1,2, \cdots, n;$$

$$f(u_nv_1) = n+3;\ f(u_nv_j) = j+1, j = 2,3, \cdots, n;$$

So we have

$$\overline{C}(u_1) = \{n+3, n+5\};$$

When $i \equiv 0 \ mod \ 2$, $\overline{C}(u_i) = \{1, i+1\}, i = 2,3,\cdots,n-1$;
When $i \equiv 1 \ mod \ 2$, $\overline{C}(u_i) = \{2, i+1\}, i = 2,3,\cdots,n-1$;

$$\overline{C}(u_n) = \{n+2, n+4\}; \overline{C}(v_1) = \{2, n+2\}; \overline{C}(v_2) = \{n+3, n+4\};$$

When $i \equiv 0 \ mod \ 2$, $\overline{C}(v_i) = \{i, n+4\}, i = 3,4,\cdots,n-1$;
When $i \equiv 1 \ mod \ 2$, $\overline{C}(v_i) = \{i, n+1\}, i = 3,4,\cdots,n-1$;

$$\overline{C}(v_n) = \{1, n\}.$$

Obviously, the f is a (n+5)-VDTC of $C_n \vee C_n$, the conclusion is true.
From all of above, the conclusion is true.

Acknowledgement

This studywas supported by Lanzhou City University Ph.D. Research Fund (LZCU-BS2013-09 and LZCU-BS2013-12).

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