On the Vertex-Distinguishing Edge Chromatic Number of $P_m \vee C_n$

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Abstract. A proper edge coloring of graph $G$ is called equitable adjacent strong edge coloring if colored sets from every two adjacent vertices incident edge are different, and the number of edges in any two color classes differ by at most one, which the required minimum number of colors is called the adjacent strong equitable edge chromatic number. In this paper, we present the edge coloring of join-graphs about path and cycle, and gain the vertex-distinguishing edge chromatic number of $P_m \vee C_n$.

Introduction

The problem about vertex-distinguishing edge coloring of $G$ is a widely used and extremely difficult problem\cite{1-4}. In \cite{5} introduced the vertex-distinguishing edge coloring of graph, and give the correspondence conjecture.

**Definition 1** $G$ is a simple graph and $k$ is a positive integer, if it exists a mapping $f : E(G) \to \{1,2,\cdots,k\}$, and satisfied with $f(e) \neq f(e')$ for adjacent edge $e,e' \in E(G)$, then $f$ is called a proper edge coloring of $G$, is abbreviated $k-PEC$ of $G$, and is called the Edge Chromatic Number of $G$.

$$\chi'(G) = \min\{k| k-PEC of G\}$$

**Definition 2**\cite{1-4} For the proper edge coloring $f$ of simple graph, if it is satisfied with $C(u) \neq C(v)$ for $V(G)(u \neq v)$, where $C(u) = \{f(uv)| uv \in E(G)\}$, then $f$ is called the Vertex-distinguishing Edge Coloring, is abbreviated $k-VDEC$ of $G$, and

$$\chi'_{vd}(G) = \min\{k| k-VDEC of G\}$$

is called the Vertex-distinguishing Edge Chromatic Number of $G$.

**Definition 3** For a graph $G$, $n_i$ is the vertex number which degree is $i$, using $\delta, \Delta$ denoted the minimum, maximum degree of $G$, it is called

$$\mu(G) = \max\left\{\min\left\{\lambda| \left(\begin{array}{c} \lambda \\ 2 \end{array}\right) \geq n_i\right\}, \delta \leq i \leq \Delta\right\}$$

*Combinatorial Degree of $G$.*

**Conjecture** For a connected graph $G$ and $|V(G)| \geq 3$, then

$$\mu(G) \leq \chi'_{vd}(G) \leq \mu(G) + 1,$$

The left of the conjecture is obviously true.

**Definition 4**\cite{6} Suppose $G$ and $H$ are two simple graphs which are vertex disjointed and edge disjointed,

$$V(G \vee H) = V(G) \cup V(H),$$

$$E(G \vee H) = E(G) \cup E(H) \cup \{uv| u \in V(G), v \in V(H)\},$$

then $G \vee H$ is called a Join-graph of $G$ and $H$.

$P_1 \vee K_n = K_{n+1}$, $P_2 \vee K_n = K_{n+2}$ has been discussed In \cite{5}, and $P_m \vee K_3 = P_m \vee C_3$ has also been discussed in another paper. Let $m \geq 3, n \geq 4$ in this paper.
In this paper, we present the edge coloring of $P_m \lor C_n$ about path $P_m$ and cycle $C_n$. The terms and signs we use in this paper but not denoted can be found in [1].

**Adjacent Strong Edge Coloring of $P_m \lor C_n$**

$P_1 \lor C_3 = K_4$ (Complete graph with order 4), $P_2 \lor C_3 = K_5$, so we have the theorem

**Theorem 1** $\chi'_{vd}(P_1 \lor C_3) = \chi'_{vd}(P_2 \lor C_3) = 5$

**Theorem 2** For $n \geq 4$, then $\chi'_{vd}(P_1 \lor C_n) = n$

**Proof** When $n \geq 4$, we have $\mu(P_1 \lor C_n) = n$. In order to prove the result is true, we only need to prove $P_1 \lor C_n$ exists $n - VDEC$.

Suppose $V(P_1) = \{u\}, C_n = v_1v_2 \cdots v_nv_1; C = \{1,2,\cdots,n\}$.

Suppose $f$ are

- $f(uv_i) = i, i = 1,2,\cdots,n$;
- $f(v_1v_{i+1}) = i + 2, i = 1,2,\cdots,n - 1$;
- $f(v_nv_1) = 1$;
- $f(v_nv_i) = 2$.

For $f$, we have

- $C(u) = \{1,2,\cdots,n\}$;
- $C(v_i) = \{i,i + 1,i + 2\}, i = 1,2,\cdots,n - 2$;
- $C(v_{n-1}) = \{n - 1,n,1\}$;
- $C(v_n) = \{n,1,2\}$.

So, the mapping $f$ is $n - VDEC$ of $P_1 \lor C_n$.

**Theorem 3** For $m \geq n$, then $\chi'_{vd}(P_m \lor C_n) = \begin{cases} m + 3, & \text{for } m > n \\ m + 4, & \text{for } m = n \end{cases}$

**Proof** For $m \geq n$, we have $\mu(P_m \lor C_n) = \begin{cases} m + 3, & \text{for } m > n \\ m + 4, & \text{for } m = n \end{cases}$.

In order to proof the theorem is true, we need to prove $P_m \lor C_n$ exists $\mu(P_m \lor C_n) - VDEC$ only.

Suppose $P_m = u_1u_2 \cdots u_m, C_n = v_1v_2 \cdots v_nv_1$

**Case 1** if $m > n$, $C = \{1,2,\cdots,m + 2,0\}$, $\tilde{C}(v) = C \setminus C(v)$.

Let $f$ et are

- $f(v_1u_j) = i + j - 1(mod(m + 3)), i = 1,2,\cdots,n - 1; j = 1,2,\cdots,m$;
- $f(v_1v_{i+1}) = i - 1, i = 1,2,\cdots,n$;
- $f(v_nv_1) = m + 2$;
- $f(v_nv_j) = n + j - 1(mod(m + 2)), j = 1,2,\cdots,m$;
- $f(u_iu_{i+1}) = n + i + 1(mod(m + 2)), i = 1,2,\cdots,m$;

For $f$,

$\tilde{C}(v_i) = \{m + i(mod(m + 3)), i = 1,2,\cdots,n - 1$.
\[ \hat{C}(v_i) = \{n - 1\}; \]
\[ C(u_1) = \{1, 2, \ldots, n, n + 2\}; \]
\[ C(u_n) = \{m, m + 1, m + 2, 1, 2, \ldots, n - 3\}; \]
\[ C(u_i) = \{i, i + 1, \ldots, i + n, i + n + 1\}(mod(m + 2)), i = 2, 3, \ldots, m - 1. \]

So, \( f \) is \((m + 3) - VDEC\) of \( P_m \lor C_n \).

**Case 2** if \( m = n \),

Let \( f \) are
\[
\begin{align*}
& f(v_iu_j) = i + j - 1(\text{mod}(n + 4)), i = 1, 2, \ldots, n - 1; j = 1, 2, \ldots, n; \\
& f(v_nu_j) = n + j - 1(\text{mod}(n + 3)), j = 1, 2, \ldots, n; \\
& f(v_iv_{i+1}) = i - 1, i = 1, 2, \ldots, n - 1; \\
& f(v_nv_1) = n + 3; \\
& f(u_iu_{i+1}) = n + i + 1(\text{mod}(n + 3)), i = 1, 2, \ldots, n - 1. 
\end{align*}
\]

Same case 1, \( f \) is \((n + 4) - VDEC\) of \( P_n \lor C_n \).

**Theorem 4** For \( n > m \geq 2 \)

\[
\chi'_{vd}(P_m \lor C_n) = \begin{cases} 
(n + 2, m = 2, 3 \\
(n + 3, m \geq 4) 
\end{cases}
\]

**Proof** For \( n > m \geq 2 \), we have

\[
\mu(P_m \lor C_n) = \begin{cases} 
(n + 2, m = 2, 3 \\
(n + 3, m \geq 4) 
\end{cases}
\]

Therefore, we need to prove \( P_m \lor C_n \) exists \( \mu(P_m \lor C_n) - VDEC \) only.

**Case 1** For \( m = 2, n \geq 4 \)

Let \( f \) are
\[
\begin{align*}
& f(v_iu_j) = i + j - 1(\text{mod}(n + 2)), i = 1, 2, \ldots, n - 1; j = 1, 2; \\
& f(v_nu_j) = n + j - 1(\text{mod}(n + 1)), j = 1, 2; \\
& f(v_iv_{i+1}) = i - 1, i = 1, 2, \ldots, n - 1; \\
& f(v_nv_1) = n + 1; f(u_1u_2) = n + 1.
\end{align*}
\]

Thus \( f \) is \((n + 2) - VDEC\) of \( P_2 \lor C_n \) obviously.

**Case 2** when \( m = 3 \)

**Case 2.1** if \( n = 4 \)

Let \( f \) are
\[
\begin{align*}
& f(u_1v_i) = i, i = 1, 2, 3, 4; \\
& f(u_2v_i) = i + 2, i = 1, 2, 3; \\
& f(u_2v_4) = 2; \\
& f(v_1v_2) = 5; \\
& f(v_2v_3) = 0; \\
& f(v_3v_4) = 1; \\
& f(v_4v_1) = 0; \\
& f(u_1u_2) = 0; \\
& f(u_2u_3) = 1.
\end{align*}
\]

Then
\[\overline{C}(u_1) = \{5\};\]
\[\overline{C}(u_3) = \{0\};\]
\[C(v_1) = \{4\};\]
\[C(v_2) = \{1\};\]
\[C(v_3) = \{2\};\]
\[C(v_4) = \{3\};\]
\[C(u_2) = \emptyset.\]

Therefore \( f \) is 6-\( VDEG \) of \( P_3 \lor C_4 \).

**Case 2.2** if \( n > 4 \)

Let \( f \) are

\[f(v_i u_j) = i + j - 1, i = 1, 2; j = 1, 2, \ldots, n;\]
\[f(u_3 v_j) = 3 + j - 1, j = 1, 2, \ldots, n - 1;\]
\[f(u_3 v_n) = 2;\]
\[f(v_1 v_2) = n + 1;\]
\[f(v_n v_1) = 0;\]
\[f(v_i v_{i+1}) = i - 2, i = 2, 3, \ldots, n - 1;\]
\[f(u_1 u_2) = 0; f(u_2 u_3) = 1.\]

so \( f \) is \( (n + 2) - VDEG \) of \( P_3 \lor C_n \).

**Case 3** For \( n > m \geq 4 \)

**Case 3.1** if \( n - 1 = m \) then

**Case 3.1.1** When \( m = 4 \), we only need to prove exist 8-\( VDEG \) of \( P_4 \lor C_5 \).

Suppose \( f \) are

\[f(u_i v_j) = i + j - 1, i = 1, 2, 3; j = 1, 2, \ldots, n - 1;\]
\[f(u_4 v_j) = 4 + j - 1, j = 1, 2, \ldots, n - 1;\]
\[f(u_4 v_n) = 3; f(v_1 v_2) = 6; f(v_2 v_3) = 7;\]
\[f(v_3 v_4) = 0; f(v_4 v_5) = 1; f(v_5 v_1) = 0;\]
\[f(u_1 u_2) = 7; f(u_2 u_3) = 0; f(u_3 u_4) = 1.\]

For \( f \), we have

\[\overline{C}(v_1) = \{5, 7\};\]
\[\overline{C}(v_5) = \{2, 4\};\]
\[\overline{C}(v_i) = \{i - 2, i - 1\}, i = 2, 3, 4;\]
\[\overline{C}(u_1) = \{0, 6\};\]
\[\overline{C}(u_4) = \{0, 2\};\]
\[\overline{C}(u_i) = \{i - 1\}, i = 2, 3.\]

Thus \( f \) is 8-\( VDEG \) of \( P_4 \lor C_5 \).

**Case 3.1.2** When \( m \geq 5 \), we only need to prove exist \( (n + 3) - VDEG \) of \( P_m \lor C_n \).

Suppose \( f \) are

\[f(u_i v_j) = i + j - 1(\text{mod}(n + 3)), i = 1, 2, 3, 5, 6 \ldots, m; j = 1, 2, \ldots, n;\]
\[f(u_4 v_j) = 4 + j - 1, j = 1, 2, \ldots, n - 1;\]
\[f(u_4 v_n) = 3;\]
\[ f(v_i v_{i+1}) = n + i + 1(mod(n + 3)), i = 2, 3, \ldots, n - 1; \]
\[ f(v_n v_1) = 0; \]
\[ f(u_i u_{i+1}) = n + i + 1(mod(n + 3)), i = 1, 2, \ldots, n - 1, \]
thus \( f \) is \((n + 3) - VDEC\) of \( P_m \lor C_n \).

**Case 3.2** if \( n > m + 1 \),

Let \( f \) are
\[ f(u_i v_j) = i + j - 1(mod(n + 3)), i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; \]
\[ f(v_i v_{i+1}) = m + i + 1(mod(n + 3)), i = 1, 2, \ldots, n - 1; \]
thus \( f \) is \((n + 3) - VDEC\) of \( P_m \lor C_n \).

Combining the case 1 to case 3, the conclusion is true.

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**References**


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