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An Analytical Study on the Differential Equation of Mesoscale Symmetric Instability under Deep Convection

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Abstract. Based on a mesoscale model with the quasi-Bossinesq approximation, we found that the disturbance stream-function would be represented by a partial differential equation, and we then obtained the analytical solution of the equation. Further, it was shown that the amplitude of instability wave changes along the direction of the characteristic line, if we consider the interaction between the change of environmental potential temperature with height and thermal wind. Besides, the interaction increases the Critical Richardson Number of Symmetric Instability $(Ri_c > Ri_{c0})$, and enlarges critical half wave-length of perturbation ($L_c^2 > L_{c0}^2$), which favors the occurrence of the Symmetric Instability (Ri_{c0} and L_{c0}^2 here are numbers when the interaction is not considered as in previous studies).

Introduction

Dynamical mechanism of mesoscale weather system is one of important applications of partial differential equation. A lot of disastrous weather events (e.g., floods, droughts, thunderstorm, hail, Tornado and so on) are the performance of mesoscale weather system. In order to reduce the impact of these disasters, mesoscale dynamics has been paid more attention and has made great progress in the past decades.

In linear symmetric instability, researchers [1-3] have done some work and obtained a lot of interesting results. As to nonlinear symmetric instability, Xu [4], Wu and Mu [5] investigated the problem of nonlinear symmetric stability in terms of different methods. A criterion of nonlinear barotropic subcritical instability was derived by Lu [6]. Itano et al. [7] has also done some work in the nonlinear symmetric instability. Among these studies, some are in shallow convection, some are in deep convection, however, even in deep convection, there is very few work considering the change of environmental potential temperature with height which exists in real atmosphere. Based on this and started from the solution of linear partial differential equations, we will discuss the influence of the interaction between this change and thermal wind on the Mesoscale Symmetric Instability using the model of deep convection.

The remainder sections of this paper are laid out as follows. In section 2, the derivation of the control equation for Mesoscale Symmetric Instability is introduced. Analytical solution of disturbance stream-function is given in Section 3. Section 4 is the influence of the interaction between the change of environmental potential temperature with height and thermal wind on the critical criterion of symmetric instability. Conclusions are presented in section 5.

Formulation

In order to study the instability mechanism of mesoscale deep convection, we use quasi- Boussinesq equations as basic equations. The set of equations includes the law of mass and energy conservation, motion in the horizontal and vertical directions. They can be written as:

$$\begin{cases} \frac{du}{dt} - fv = -\frac{1}{\rho_s}\frac{\partial \tilde{p}}{\partial x}, \\ \frac{dv}{dt} + fu = -\frac{1}{\rho_s}\frac{\partial \tilde{p}}{\partial y}, \\ \frac{dw}{dt} - g\frac{\tilde{\theta}}{\theta_s} = -\frac{1}{\rho_s}\frac{\partial \tilde{p}}{\partial z}, \\ \frac{d\theta}{dt} = Q, \\ \frac{\partial \rho_s u}{\partial x} + \frac{\partial \rho_s v}{\partial y} + \frac{\partial \rho_s w}{\partial z} = 0. \end{cases} \tag{1}$$

Here, $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, x, y, z are directed eastward, northward, and upward, respectively; and u, v, w denote respectively, zonal, meridional and vertical velocities; t is the time coordinate; $f(=2\Omega\sin\phi)$ is the Coriolis parameter (Ω denoting the Earth's rotation rate and ϕ the latitude); g is the gravity constant; Q is non-adiabatic heating; θ is potential temperature; p indicates the dynamic pressure, and $p_s(z), \rho_s(z), \theta_s(z)$ represent pressure, density, and potential temperature of environmental atmosphere respectively, what's more,

$$\tilde{p} = p - p_s(z), \quad \tilde{\theta} = \theta - \theta_s(z). \tag{2}$$

The variables include basic and perturbation field, the basic field is large-scale background and the perturbation field is mesoscale disturbance. That is,

$$u = \bar{u}(y,z) + u', v = v', w = w', \ \tilde{\theta} = \bar{\theta}(y,z) + \theta', \tilde{p} = \bar{p}(y,z) + p', \tag{3}$$

where $\bar{u}(y,z), \bar{\theta}(y,z), \bar{p}(y,z)$ are basic wind, potential temperature and pressure, respectively, which are supposed to be linear function of y, z, "'" represents disturbance, (2) and (3) are substituted into (1). Letting

$$(u, v, w, \theta) = \rho_s(u', v', w', \frac{g}{\theta_s}\theta'), \quad p = p', \quad S^2 \equiv -\frac{g}{\theta_s}\frac{\partial \bar{\theta}}{\partial y} \equiv f\bar{u}_T, \ N^2 \equiv \frac{g}{\theta_s}\frac{\partial(\bar{\theta} + \theta_s)}{\partial z}, \ f_\alpha = f - \bar{u}_y, \tag{4}$$

where $\bar{u}_T$ is thermal wind on temperature field. In addition, we consider that disturbance is symmetric about x-axis ($\frac{\partial}{\partial x} = 0$) and adiabatic ($Q = 0$), a disturbance stream-function $\psi(y, z, t)$ is introduced here defined as follows [8,9]:

$$v = \frac{\partial \psi}{\partial z}, w = -\frac{\partial \psi}{\partial y}. \tag{5}$$

We consider that geostrophic, static, and thermal wind balance

$$f\bar{u} = -\frac{1}{\rho_s}\frac{\partial \bar{p}}{\partial y}, \quad -g\frac{\bar{\theta}}{\theta_s} = -\frac{1}{\rho_s}\frac{\partial \bar{p}}{\partial z}, \quad \bar{u}_z = \bar{u}_T. \tag{6}$$

With the aid of (2) - (6), equations in (1) are reduced to the partial differential equation of ψ:

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi + ff_\alpha\frac{\partial^2\psi}{\partial z^2} + 2f\bar{u}_z\frac{\partial^2\psi}{\partial y\partial z} + N^2\frac{\partial^2\psi}{\partial y^2} - M\frac{\partial\psi}{\partial y} = 0. \tag{7}$$

Here $\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator, and represents the interaction between the change of environmental potential temperature with height and thermal wind. As to partial differential equation, Wang et al. have done lots of work [10-11]. Here we want to get the analytical solution of equation (7).

$$M = f\bar{u}_T \frac{d\ln\theta_s}{dz} \tag{8}$$

If environmental potential temperature does not change with height, that is $M=0$, then (7) becomes the equation in the previous studies [2]. However, in this paper we consider the change of θ_s with height in deep convection and the interaction between thermal wind and it.

In order to obtain the analytical solution of (7), we consider the following boundary conditions, (7) (9) are the starting equations of Mesoscale Symmetric Instability.

$$\psi(y,z,t)\big|_{z=0,H}=0,\quad \psi(y,z,t)\big|_{y=0,H} \text{ is free boundary.} \tag{9}$$

Analytical Solution of Disturbance Stream-Function

Letting and substituting (10) into (7), we have

$$\psi(y,z,t)=e^{-ay-bz+\omega t}\varphi(y,z) \tag{10}$$

$$\begin{aligned}&(ff_\alpha+\omega^2)\varphi_{zz}+(N^2+\omega^2)\varphi_{yy}+2f\bar{u}_z\varphi_{yz}-[2b(ff_\alpha+\omega^2)+2f\bar{u}_z a]\varphi_z\\&-[2a(N^2+\omega^2)+2f\bar{u}_z b+M]\varphi_y+[b^2(ff_\alpha+\omega^2)+a^2(N^2+\omega^2)+2f\bar{u}_z ab+Ma]\varphi=0.\end{aligned} \tag{11}$$

Now (11) is rewritten as follows:

$$(ff_\alpha+\omega^2)\varphi_{zz}+(N^2+\omega^2)\varphi_{yy}+2f\bar{u}_z\varphi_{yz}+E\varphi=0, \tag{12}$$

When

$$a=(ff_\alpha+\omega^2)M/2[f^2\bar{u}_z{}^2-(N^2+\omega^2)(ff_\alpha+\omega^2)],b=-f\bar{u}_zM/2[f^2\bar{u}_z{}^2-(N^2+\omega^2)(ff_\alpha+\omega^2)]. \tag{13}$$

$$E=(ff_\alpha+\omega^2)M^2/4[f^2\bar{u}_z{}^2-(N^2+\omega^2)(ff_\alpha+\omega^2)]. \tag{14}$$

At the same time, we have

$$\varphi(y,z)\big|_{z=0,H}=0,\qquad \varphi(y,z)\big|_{y=0,H} \text{ is free boundary.} \tag{15}$$

If $M=0$, we can see obviously that a,b,E are equal to 0 and (11) is consistent with the equation of Zhang [2].

Symmetric instability is a dynamical phenomenon and is a result of imbalance of gravity, pressure gradient and Coriolis Force. So symmetric instability appears as ramp flow phenomenon, we define

$$\varphi(y,z)=\phi_0\sin\frac{n\pi z}{H}\sin\frac{\pi}{L}(y-\alpha z), \tag{16}$$

where n is the vertical wave number, L is the horizontal half-wavelength and $\frac{1}{\alpha}$ is the slope of constant phase.

With the aid of (16), (12) becomes:

$$P\sin\frac{n\pi z}{H}\sin\frac{\pi}{L}(y-\alpha z)+G\cos\frac{n\pi z}{H}\cos\frac{\pi}{L}(y-\alpha z)=0, \tag{17}$$

Where

$$P=-(ff_\alpha+\omega^2)(\frac{n^2\pi^2}{H^2}+\frac{\alpha^2\pi^2}{L^2})+2f\bar{u}_z\frac{\alpha\pi^2}{L^2}-(N^2+\omega^2)\frac{\pi^2}{L^2}+E,\quad G=-2(ff_\alpha+\omega^2)\frac{n\alpha\pi^2}{HL}+2f\bar{u}_z\frac{n\pi^2}{HL}. \tag{18}$$

Then the equation of (17) is multiplied by $\sin\frac{n\pi z}{H}\sin\frac{\pi}{L}(y-\alpha z)$ and integrated in the region of $0\le z\le H, 0\le y\le Y$, it is showed that P is equal to zero.

When L satisfies that

$$L=L_m=\frac{1}{m}Y(m=1,2,3\cdots), \tag{19}$$

Then

$$G=0,\ \alpha=f\bar{u}_z/(ff_\alpha+\omega^2),\ b=-a\alpha. \tag{20}$$

Substituting (16) into (12), we get frequency equation

$$(ff_\alpha+\omega^2)^2(n^2\pi^2/H^2)+[(ff_\alpha+\omega^2)(N^2+\omega^2)-f^2\bar{u}_z^{\,2}](\pi/L_m)^2-(ff_\alpha+\omega^2)E=0. \tag{21}$$

When ω is a real number, that is, only stationary instability waves, then is showed obviously that wave amplitude changes along the direction of characteristic line.

$$\psi(y,z,t)=\varphi_0 e^{-a(y-\alpha z)+\omega t}\sin(n\pi z/H)\sin[(\pi/L_m)(y-\alpha z)] \tag{22}$$

If we do not consider the term of M, that is $M=0$, the solution is consistent with that of Zhang[2].

The Influence of the Interaction between the Change of Environmental Potential Temperature with Height and the Thermal Wind on Symmetric Instability

The Critical Half-Wave Length of Perturbation (L_c)

In order to obtain critical value, we let that ω is equal to 0. From (21), we have

$$L_m^{\,2}=L_c^{\,2}=[(1/n^2)L_0^{\,2}(f/f_\alpha-Ri)]/\{(f/f_\alpha)[1-H^2M^2/4\pi^2n^2ff_\alpha\bar{u}_z^{\,2}(f/f_\alpha-N^2/\bar{u}_z^{\,2})]\}, \tag{23}$$

where L_0 is the radius of inertia circle and is equal to $H\bar{u}_z/f$. But when $M=0$, then

$$L_{c0}^{\,2}=[(1/n^2)L_0^{\,2}(f/f_\alpha-N^2/\bar{u}_z^{\,2})]/(f/f_\alpha). \tag{24}$$

Comparing (23) with (24), we can see that $L_c^{\,2}>L_{c0}^{\,2}$. So, the interaction between the change of environmental potential temperature and thermal wind makes the critical half-wave length of perturbation larger, and makes Symmetric Instability occur more easily.

Critical Richardson Number (Ri) of Symmetric Instability

Same as the critical half-wave length of perturbation above, Ri is critical value when ω is equal to 0.The last term of frequency equation (21) can be approximated as follows:

$$-(ff_\alpha+\omega^2)E\approx(ff_\alpha+\omega^2)^2M^2/4f^2\bar{u}_z^{\,2}. \tag{25}$$

It is obviously that this approximation underestimates the interaction between the change of environmental potential temperature and thermal wind.

Substituting (25) into (21), we get the Critical Richardson Number of Symmetric Instability from the approximation frequency equation (21):

$$Ri\equiv\frac{N^2}{\bar{u}_z^{\,2}}<\frac{f}{f_\alpha}-\frac{f_\alpha}{f}(\frac{L_m}{L_0})^2n^2+\frac{ff_\alpha L_m^{\,2}M^2}{4f^2\pi^2\bar{u}_z^{\,4}}\equiv Ri_c. \tag{26}$$

When $M=0$, then (27) is the criterion of the Critical Richardson Number of Symmetric Instability which is obtained by Zhang [2].

$$Ri \equiv \frac{N^2}{\bar{u}_z^{\,2}} < \frac{f}{f_\alpha} - \frac{f_\alpha}{f}(\frac{L_m}{L_0})^2 n^2 \equiv Ri_{c0}. \tag{27}$$

It is obviously showed that $Ri_c > Ri_{c0}$ by comparing (26) with (27). Therefore, the interaction makes the Critical Richardson Number of Symmetric Instability increased, which favors the occurrence of Symmetric Instability.

Conclusions

In this paper, we can see that partial differential equation plays a very important role in Mesoscale meteorology. Considering the change of environmental potential temperature with height under deep convection which exits in real atmosphere, we deduced that the equation of disturbance stream-function is a partial differential equation by use of the quasi-Bossinesq approximation model. After analyzing the solution of the equation, the critical criterion of Symmetric Instability was obtained. The results show that:

(1) The interaction between the change of environmental potential temperature with height and thermal wind has an important role on Mesoscale Symmetric Instability, it makes the non-propagation wave of Symmetric Instability change, wave amplitude changes along the direction of characteristic line in accordance with e-index;

(2) The interaction increases the Critical Richardson Number of Symmetric Instability $(Ri_c > Ri_{c0})$, and makes Symmetric Instability occur more easily;

(3) The interaction enlarges critical half wave-length of perturbation ($L_c^{\,2} > L_{c0}^{\,2}$), which improvements the result of Zhang[2] to a certain extent.

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