A Perturbed Iterative Algorithm for Split General Mixed Variational Inequality Problem

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Keywords: Split general mixed variational inequality problem, Split general quasi-variational inequality problem, Perturbed iterative algorithm, Convergence.

Abstract. In this paper, we introduce a split general mixed variational inequality problem, which is a natural extension of a split variational inequality problem, split general quasi-variational inequality problem in Hilbert spaces. Using the resolvent operator technique, we propose a perturbed iterative algorithm for the split general mixed variational inequality problem. Further, we discuss the convergence criteria of the iterative algorithm. The results presented in this paper extend and improve many previously known results in this area.

Introduction

In a recent paper [1], Kazmi has developed an iterative algorithm for finding approximate solution for a new split general quasi-variational inequality problem in Hilbert spaces. The aim of this work is to extend his idea to more general problem. Throughout the paper unless stated otherwise, for each \(i \in \{1,2\}\) let \(H_i\) be a real Hilbert space with inner product \(\langle \ , \ \rangle\) and norm \(\|\ |\|\), let \(f_i : H_i \to H_i\), \(g_i : H_i \to H_i\) be continues mappings with \(\text{Im} \ g_i \cap \text{dom} \partial \neq \emptyset\), where \(\partial \varphi_i\) denotes the subdifferential of a proper, convex and lower semicontinuous function \(\varphi_i : H \to \mathbb{R} \cup \{+\infty\}\). Let \(A : H_1 \to H_2\) be a bounded linear operator with its adjoint operator \(A^*\). we consider the following problem: Find \(x_1^* \in H_1\), such that \(g_i(x_1^*) \in \text{dom} \partial_i\) and

\[
\langle f_i(x_i^*), x_i \rangle - \langle g_i(x_i^*) \rangle \geq \langle i(x_i^*) \rangle, \forall x_i \in H_i, \tag{1}
\]

and such that \(x_2^* = Ax_1^* \in H_2, g_2(x_2^*) \in \text{dom} \partial_2\) solves

\[
\langle f_2(x_2^*), x_2 \rangle - \langle g_2(x_2^*) \rangle \geq \langle 2(x_2) \rangle, \forall x_2 \in H_2. \tag{2}
\]

We call problem (1) - (2) the split general mixed variational inequality problem (in short, SpGMVIP). SpGMVIP (1) - (2) amounts to saying: find a solution of general mixed variational inequality problem (1) image under a given bounded linear operator is a solution of general mixed variational inequality problem (2). For convenience, we denote the solution set of SpGMVIP (1) - (2) by \(\Gamma = \{x_1^* \in H_1 | x_1^* \text{solves (1)} \text{ and } Ax_1^* \in H_2 \text{ solves (2)}\}\). If we set \(i(\ ) = \delta c_i(\ m_i(x_i^*)) = \delta c_i + m_i(x_i^*)(\ )\), \(m_i : H_i \to H_i\) is a single-valued mapping, where \(C_i(x_i^*) = C_i + m_i(x_i^*)\), and \(C_i\) is a closed convex subset of \(H_i\), then SpMVIP (1) - (2) is reduced to the following split general quasi-variational inequality problem (in short, SpGQVIP): Find \(x_1^* \in H_1\), such that \(g_i(x_1^*) \in C_i(x_1^*)\) and

\[
\langle f_i(x_i^*), x_i \rangle - \langle g_i(x_i^*) \rangle \geq 0, \forall x_i \in C_i(x_i^*), \tag{3}
\]
and such that \( x_2^* = Ax_1^*, \) \( g_2(x_2^*) \in C_2(x_2^*) \) solves
\[
\left( f_2(x_2^*), x_2, g_2(x_2^*) \right) \geq 0, \quad \forall \ x_2 \in C_2(x_2^*). \tag{4}
\]

This problem was introduced and studied by Kazmi in [1] and he exhibited split quasi-variational inequality problem, split general variational inequality problem and quasi-variational inequality problem as special cases of SpGQVIP (3)-(4). For details, see reference [1]. It is worth noting that SpGMVIP (1)-(2) is quite general and includes as special cases split minimization between two spaces so that the image of a minimize of a given function, under a bounded linear operator, is a minimizer of another function; split zero problem and the split feasibility problem which have already been studied and used in practice as a model in the intensity-modulated radiation therapy planning, see [2-5]. In aword, SpGMVIP (1)-(2) is more general, which is one of our motivations to write this paper. Using the resolvent operator technique about the maximal monotone mapping, we propose a perturbed iterative algorithm for SpGMVIP (1)-(2) and discuss the convergence criteria of the iterative algorithm. The result presented here extends and improves the previously known results of this area.

**Perturbed Iterative Algorithms**

To begin with, let us transform SpGMVIP (1)-(2) into fixed point problem.

**Lemma 2.1.** \( x_1^* \in \Gamma \) if and only if \( x_1^* \) satisfies the following relations
\[
g_1(x_1^*) = J_{\rho_1}^{\beta_1}(g_1(x_1^*) \rho_1f_1(x_1^*)), \tag{2.1a}
\]
\[
g_2(Ax_1^*) = J_{\rho_2}^{\beta_2}(g_2(Ax_1^*) \rho_2f_2(Ax_1^*)). \tag{2.1b}
\]

Where \( \rho_1 > 0 \) is a constant and \( J_{\rho_1}^{\beta_1} := (I + \rho_1 \partial \varphi_1)^{-1} \) is the resolvent operator of the maximal monotone mapping \( \partial \varphi_1 \).

Based on Lemma 2.1, we can propose the following perturbed iterative algorithms for approximating a solution to SpGMVIP (1.1a) - (1.1b). let \( \{\alpha^n\} \subseteq (0,1) \) be a sequence such that \( \sum_{n=0}^{\infty} \alpha^n = + \infty \), and let \( \rho_1, \rho_2, \gamma \) be the parameters with positive values.

**Algorithm 2.1.** Given \( x_1^0 \in H_1 \), compute the iterative sequence \( \{x_i^n\} \) defined by the iterative schemes
\[
g_1(y^n) = J_{\rho_1}^{\beta_1}(g_1(x_1^n) \rho_1f_1(x_1^n)), \tag{2.2a}
\]
\[
g_2(z^n) = J_{\rho_2}^{\beta_2}(g_2(Ay^n) \rho_2f_2(Ay^n)), \tag{2.2b}
\]
\[
x_i^{n+1} = (1 - \alpha^n)x_i^n + \alpha^n[g^n + \gamma A^*(z^n - Ay^n)] + \alpha^n e_n, \tag{2.2c}
\]

for all \( n = 0,1,2,\ldots, \rho_1, \rho_2, \gamma > 0 \), and take into account a possible inexact computation, an error \( e_n \) is added in the righthand side of (2.2c). Moreover, we consider other perturbations by replacing in (2.2a) and (2.2b), by \( n \), where the sequence \( \{\varphi_n\} \) approximates \( \varphi_1 \), \( \{\alpha^n\} \) is a collection of proper convex semi-continuous functions on \( H \).

In order to obtain our main results, we need the following definition, assumption end lemmas.

**Definition 2.1.** A nonlinear mapping \( f : H_1 \to H_1 \) is said to be
(i) \( \alpha \)-strongly mouotone if there exists a constant \( \alpha > 0 \) such that
\[
\langle f_0(x), f_0(y) - f_0(x), y - x \rangle \geq \alpha \|x - y\|^2, \forall x, y \in H_1,
\]
(ii) \( \beta \)-Lipschitz continuous if there exists a constant \( \beta > 0 \) such that
\[
\|f(x) - f(y)\| \leq \beta \|x - y\|, \forall x, y \in H_1.
\]
Remark 2.1. It is easy to know that if \( f : H_1 \to H \) is \( \alpha \)-strongly monotone and \( \beta \)-Lipschitz continuous, then \( \alpha \leq \beta \).

Assumption 2.1. For \( i \in \{1, 2\} \), let \( \varphi : H_i \to \mathbb{R} \cup \{+\infty\} \) be a proper, convex and lower semicontinuous function, which \( \{\varphi_i^n\} \) approximate \( \varphi_i \) and satisfies the condition:
\[
\lim_{n \to \infty} \left\| J_{\rho_i}^{\varphi} (v_i) - J_{\rho_i}^{\varphi_i} (v_i) \right\| = 0, \forall v_i \in H_i.
\]

Lemma 2.2 (see [6, Lemma 2.17]). Let \( \{a_k\} \) be a sequence of nonnegative real numbers satisfying the condition
\[
a_{k+1} \leq (1 - m_k)a_k + m_k\delta_k, \forall k \geq 0.
\]
Where \( \{m_k\}, \{\delta_k\} \) are sequences of real numbers such that
(i) \( \{m_k\} \subseteq [0, 1] \) and \( \sum_{k=0}^{\infty} m_k = \infty \), or, equivalently, \( \prod_{k=0}^{\infty} (1 - m_k) := \lim_{k \to \infty} \prod_{j=0}^{k} (1 - m_j) = 0 \);
(ii) \( \limsup_{k \to \infty} \delta_k \leq 0 \), or \( \sum_{k=0}^{\infty} \delta_k m_k \) is convergent. Then, \( \lim_{k \to \infty} a_k = 0 \).

Lemma 2.3. Let \( H \) be a real Hilbert space, for all \( x, y \in H \), the following formulas hold:
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \|x + y\|^2 = \|y\|^2 + 2\langle x, y \rangle + \|y\|^2.
\]

Main Results

Now, we study the convergence of Algorithm 2.1 for SpGMVIP (1.1a)-(1.1b).

Theorem 3.1. For each \( i \in \{1, 2\} \), let \( g_i : H_i \to H_i \) be \( \delta_i \)-Lipschitz continuous such that \( (g_i, I_i) \) is \( \delta_i \)-strongly monotone, where \( I_i \) is the identity operator on \( H_i \). Let \( f_i : H_i \to H_i \) be \( a_i \)-strongly monotone with respect to \( g_i \) and \( \beta_i \)-lipschitz continuous. Let \( A : H_1 \to H_2 \) be a bounded linear operator and let \( A^* \) be its adjoint operator. Suppose \( x_i^* \in P \) and Assumption 2.2 holds. Then the sequence \( \{x_i^k\} \) generated by Algorithm 2.1 converges strongly to \( x_i^* \) provided that the constant \( \rho_i \) and \( \gamma \) satisfy the conditions
\[
\rho_i \frac{\alpha}{\beta_i^2} < \sqrt{\tau_i^2 + \gamma_i^2} \quad \text{and} \quad \beta_i \gamma < \sqrt{\tau_i^2 + \gamma_i^2},
\]
where \( \rho_i \), \( \alpha \), \( \beta_i \) and \( \gamma \) are defined by (3.1).

Proof. Since \( x_i^* \in \Gamma \), then \( x_i^* \in H_i \) such that \( g_i(x_i^*) = \text{dom} \Phi_i (g_i(x_i)) \) and
\[
g_i(x_i^*) = J_{\rho_i}^{g_i} (g_i(x_i^*)) = J_{\rho_i}^{g_i} (g_i(x_i^*) \rho_i f_i(x_i^*)), \quad (3.1)
\]
\[
g_i(x_i^*) = J_{\rho_i}^{g_i} (g_i(x_i^*)) \rho_i f_i(x_i^*).
\]
for \( \rho_i > 0 \) and \( x_i^* = Ax_i^* \). From Algorithm 2.1 (2.2a), Assumption 2.1 and (3.1), we have
\[
\left\| g_i(y_i^*) - g_i(x_i^*) \right\| \leq \left\| J_{\rho_i}^{\varphi_i} (g_i(x_i^*)) - J_{\rho_i}^{\varphi_i} (g_i(y_i^*)) \right\| \leq \left\| J_{\rho_i}^{\varphi_i} (g_i(x_i^*)) - J_{\rho_i}^{\varphi_i} (g_i(y_i^*)) \right\| \quad (3.3)
\]
\[
\| J_{\rho_i}^{\varphi_i} (g_i(x_i^*)) - J_{\rho_i}^{\varphi_i} (g_i(y_i^*)) \| + \| J_{\rho_i}^{\varphi_i} (g_i(x_i^*)) - J_{\rho_i}^{\varphi_i} (g_i(y_i^*)) \|.
\]
where \( \varepsilon_1^n = \| J_{\rho_1}^n (g_1(x^n_1), \rho_1 f_1(x^n_1)) - J_{\rho_1}^{n-1} (g_1(x^n_1), \rho_1 f_1(x^n_1)) \| \) and \( \lim_{n \to \infty} \varepsilon_1^n = 0 \) owns to Assumption 2.1. Now, using the facts that \( f_i \) is \( \alpha_i \)-strongly monotone with respect to \( g_i \) and \( \beta_i \)-lipschitz continuous, and \( g_i \) is \( \delta_i \)-Lipschitz continuous, we have

\[
\| g_1(x^n_i) \| \leq 2 \rho_i \alpha_i + \rho_i^2 \beta_i^2 \| x^n_i \| + \varepsilon_1^n.
\]

Combining (3.3) and (3.4), we have

\[
\| g_1(y^n) \| \leq \sqrt{\delta_i^2 + 2 \rho_i \alpha_i} + \varepsilon_1^n.
\]

Since \( (g_i, I_i) \) is \( \sigma_i \)-strongly monotone, we have

\[
\| y^n - x^n_i \|^2 \leq 2 \rho_i \alpha_i + \rho_i^2 \beta_i^2 \| x^n_i \| + \varepsilon_1^n,
\]

which implies

\[
\| y^n - x^n_i \|^2 \leq \tau_i \| g_1(y^n) \| - g_1(x^n_i) \|,
\]

where \( \tau_i = \frac{1}{\sqrt{1 + 2 \sigma_i}} \). From (3.5) and (3.6), we have

\[
\| y^n - x^n_i \|^2 \leq \theta_1 \| x^n_i \| + \tau_i \varepsilon_1^n,
\]

where \( \theta_1 = \tau_i \sqrt{\delta_i^2 + 2 \rho_i \alpha_i} \). Similarly, from Algorithm 2.1 (2.2b), Assumption 2.1 and (3.2) and using the facts that \( f_2 \) is \( \alpha_2 \)-strongly monotone with respect to \( g_2 \) and \( \beta_2 \)-Lipschitz continuous, \( (g_2, I_2) \) is \( \sigma_2 \)-strongly monotone, and \( g_2 \) is \( \delta_2 \)-Lipschitz continuous, we have

\[
\| g_2(z^n) \| \leq \sqrt{\delta_i^2 + 2 \rho_2 \alpha_2} + \varepsilon_2^n,
\]

and

\[
\| z^n - A x_i^n \|^2 \leq \theta_2 \| y^n - Ax_i^n \|^2 + \tau_2 \varepsilon_2^n,
\]

where \( \tau_2 = \frac{1}{\sqrt{2 \sigma_2 + 1}} \). And \( \lim_{n \to \infty} \varepsilon_2^n = 0 \) owns to Assumption 2.1

\[
\| x_i^{n+1} \| \leq (1 - \alpha^n \| x_i^n \|) + \alpha^n \| y^n - Ax_i^n \|^2 + \gamma \| A^*(z^n - Ax_i^n) \|^2 + \alpha_i \| x_i^n \|^2.
\]

Further, using the definition of \( A^* \), the face that \( A^* \) is a bounded linear operator with \( \| A^* \| = \| A \| \), and the given condition on \( \gamma \), we have

\[
\| y^n - x_i^n \|^2 \leq (1 - \alpha^n \| x_i^n \|) + \alpha^n \| y^n - Ax_i^n \|^2 + \gamma \| A^*(z^n - Ax_i^n) \|^2 + \alpha_i \| x_i^n \|^2.
\]

and using (3.9), we have

\[
\| A^*(z^n - Ax_i^n) \|^2 \leq \| A^* \|^2 \| z^n - Ax_i^n \|^2 \leq \theta_2 \| A \| \| y^n - Ax_i^n \|^2 + \| A \| \tau_2 \varepsilon_2^n.
\]
It follows from (3.10)-(3.12), we obtain
\[ \|x_{i+1}^n - x_i^n\| \leq (1 - a_n)\|x_i^n - x_{i+1}^n\| + a_n(1 + \gamma\|A\|^2\|\theta_2\|)\|x_i^n - x_{i+1}^n\| \\
+ a_n(1 + \gamma\|A\|^2\|\theta_2\|)\|\tau_1\epsilon_i^n + \gamma\|A\|\|\tau_2\epsilon_2^n + \|\epsilon_n\|\| \\
= \left[1 - a_n(1 - 0)\|x_i^n - x_{i+1}^n\| + a_n(1 + \gamma\|A\|^2\|\theta_2\|)\|\tau_1\epsilon_i^n + \gamma\|A\|\|\tau_2\epsilon_2^n + \|\epsilon_n\|\| \right]^n \] (3.13)

where \(\theta = 0, (1 + \gamma\|A\|^2\|\theta_2\|)\). It follows from the conditions on \(\rho_1, \rho_2\) and \(\gamma\) that \(\theta \in (0,1)\). Set \(a_n = \|x_i^n - x_{i+1}^n\|, m_n = a_n(1 - 0), \delta_n = 1 - \frac{1}{1 + \gamma\|A\|^2\|\theta_2\|} + \|A\|\|\tau_2\epsilon_2^n + \|\epsilon_n\|\|, \forall n \geq 0\). By virtue of (3.13), we have \(a_{n+1} \leq (1 - m_n)a_n + m_n\delta_n\). Moreover, Conditions (i) and (ii) of Lemma 2.2 are all satisfied. It follows that \(\{x_i^n\}\) converges strongly to \(x_i^\ast\) as \(n \to \infty\). Since \(A\) is continuous, it follows from (3.5), (3.7), (3.8), (3.9) that \(y^n \to x_i^\ast, g_1(y^n) \to g_1(x_i^\ast), Ay^n \to Ax_i^\ast, z^n \to x_i^\ast\) and \(g_2(z^n) \to g_2(Ax_i^\ast)\) as \(n \to \infty\). This completes the proof.

**Remark 3.1.** Theorem 3.1 extends the corresponding results of [1].

**Acknowledgements**

This work is supported by the National Natural Science Foundation of China (11371070).

**References**


