Strategies on Direct Kinematics to Generalized Stewart Platforms

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Abstract. Stewart platform is a parallel linkage, which is widely used in the area of parallel robotics. Generalized Stewart Platform (GSP) consists of two rigid bodies connected with six constraints between six pairs of geometric primitives in the base and the moving platform respectively. Direct kinematics to GSP is to determine the pose of moving platform according to six constraints between the base and the moving platform. Strategies on direct kinematics to GSP were presented, which were based on graph theory and symbolic computation method. A GSP can be decomposed into smaller ones if possible, and the corresponding parameters conditions of real solutions to direct kinematics to the GSP can be obtained according to these strategies.

Introduction

The Stewart platform, originated from the mechanism designed by Stewart for flight simulation [1] is a parallel manipulator consisting of two rigid bodies: a moving platform, or simply a platform, and a base. The position and orientation (pose) of the base are fixed. The base and the platform are connected with six extensible legs. A large portion of the work on Stewart platform is focused on the direct kinematics: determining the pose of the platform for a given set of lengths of the legs. This problem is still not solved completely until now. To find new and more practical parallel mechanisms for various purposes, the generalized Stewart platform (abbr. GSP) consisting of two rigid bodies connected with six distance and/or angular constraints between six pairs of points, lines and/or planes in the base and moving platform respectively is introduced, which could be considered as the most general form of parallel manipulators with six degrees of freedom [2]. The number of parameters needed to fix a geometric primitive is called the degree of freedom of the primitive (abbr. DOF). The number of scalar equations corresponding to a constraint is called the degree of the constraint (abbr. DOC). Because in the space a rigid body has six DOFs, i.e. three rotational DOFs and three translational DOFs, the total DOCs needed to determine the pose of a rigid body is six. In [2], Assuming that DOC of each constraint is one, the GSPs are divided into four classes, i.e. 3D3A, 4D2A, 5D1A and 6D according to the constraints added, where A represents angular constraint and D represents distance constraint. For example, 3D3A means there are three angular and three distance constraints between the base and the platform.

There are 3850 types of GSPs totally, where 3D3A GSPs are 1120. In [2], the closed-form solutions to 3D3A GSPs are given by imposing three angular constraints to get the rotation matrix firstly and imposing three distance constraints to get the translation matrix secondly. The best maximal numbers of real solutions for these GSPs may be obtained from these solutions.

The planar generalized Stewart platform consists of two rigid bodies connected by three distance and/or angular constraints between three pairs of points and/or lines in the base and the moving platform respectively. There are 16 types of planar GSPs totally. In [3], the classification of direct kinematics to these planar GSPs is discussed. The problem of finding direct kinematics to the planar GSP is transformed into solving nonlinear polynomial equation system. The parameter conditions of obtaining real solutions to 16 types of planar GSPs are presented with Wu-Ritt characteristic method [4] and complete discrimination system for polynomials [5].
In this paper, we discuss the strategies on direct kinematics to spatial GSP with graph theory based on the analysis of degrees of freedom of geometric primitives and degrees of constraints, Wu-Ritt characteristic method and complete discrimination system for polynomials.

**Preliminary Concepts**

We will use $p$, $l$, and $h$ to represent points, lines, and planes respectively. The angular and distance constraints between two primitives $v_1$ and $v_2$ are denoted by $\text{ANG}(v_1, v_2)$ and $\text{DIS}(v_1, v_2)$, respectively. The degrees of constraints $\text{ANG}(v_1, v_2)$ and $\text{DIS}(v_1, v_2)$ are denoted by $\text{DOC}(\text{ANG}(v_1, v_2))$ and $\text{DOC}(\text{DIS}(v_1, v_2))$ respectively. The degrees of angular and distance constraints considered in this paper are as follows.

1. $\text{DOC}(\text{ANG}(l_1, l_2)) = 2$, if lines $l_1$ and $l_2$ are parallel. Otherwise, $\text{DOC}(\text{ANG}(l_1, l_2)) = 1$.
2. $\text{DOC}(\text{ANG}(l, h)) = 2$, if line $l$ is perpendicular to plane $h$. Otherwise, $\text{DOC}(\text{ANG}(l, h)) = 1$.
3. $\text{DOC}(\text{DIS}(l, h)) = 2$, which means that line $l$ has to be parallel to plane $h$.
4. $\text{DOC}(\text{DIS}(l_1, l_2)) = 3$, if lines $l_1$ and $l_2$ are parallel and non-coincidence.
5. $\text{DOC}(\text{DIS}(p_1, p_2)) = 1$ and $\text{DOC}(\text{DIS}(p_1, p_2)) = 1$, if points $p_1$ and $p_2$ are non-coincidence.

The degrees of coincidence constraint between two points, two lines, and two planes are three, four, and three, respectively.

**GSP Constraint Graph**

We use a bipartite graph $G = (V, C, E)$ to represent the problem of direct kinematics to a GSP, where $V$ and $C$ are the vertex sets of $G$, $E$ is the edge set. The details are as follows.

1. The vertices in $V$ represent the geometric primitives in the moving platform involved in the constraints between the base and the platform, called primitive vertices. The minimal number of vertices in $V$ is two and the maximal number of vertices in $V$ is six.
2. The vertices in $C$ represent the constraints, called constraint vertices. A constraint vertex between two primitives in $V$ is called an internal constraint. A constraint vertex between the base and the platform is called an external constraint.
3. The edges in $E$ is defined as $\{c, v\} \in V, c \in C\}$, where $c$ is a constraint involving $v$. The direction of each edge is always directed from $v$ to $c$. If the constraint vertex is an external constraint, the direction of the edge is also directed from $c$ to $v$. For each primitive vertex, the minimal number of degree of external constraint is one.
4. For a vertex $v \in V$, the sum of DOCs directed to $v$ is less than or equal to DOF(v). If the sum of DOCs directed to $v$ is equal to DOF(v), the vertex is called a saturated vertex.
5. If there are five or six vertices in $V$, each vertex is unsaturated.

**Classification of GSPs Based on Constraint Graphs**

The GSPs that may contain saturated vertex are classified into three types according the number of primitive vertices in the constraint graph.

1. There are two primitive vertices.

   According to rigid bodies analysis in [6], these two primitives are: a point and a line, where the point and the line are non-coincidence; or a plane and a line, where the plane and the line are neither parallel nor perpendicular to each other. The corresponding constraint graphs are shown as Figure 1(a) and (b), where the circle represents the constraint vertex and the square represents the primitive vertex. Then we will use well-constrained completion and decomposition method [7] to the GSPs to find the construction sequences. The well-constrained completion constraint graphs are shown as Figure 2(a) and (b). For constraint graph in Figure 2(a), we will fix the point/line firstly and the line secondly.
For constraint graph in Figure 2(b), we can fix the line firstly and the point/plane secondly.

(2) There are three primitive vertices.

According to rigid bodies analysis in [6], there are ten cases. In this paper, we only list constraint graphs in which the primitive vertices can be fixed one by one according to the directions of the edges shown as Figure 3.

![Figure 1. GSP containing two primitive vertices.](image1)

![Figure 2. Well-constrained completion graph in Figure 1.](image2)

(3) There are four primitive vertices.

Now, there are 15 cases without considering the types of the internal constraints in the platform. To find GSPs that can be fixed one by one or transformed into smaller ones, the sum of degrees of external constraints added to a plane or a point in the platform must be three. We will use well-constrained completion and decomposition method [7] to the GSPs to find the construction sequences.

**Strategies on Direct Kinematics to GSPs**

Strategies on direct kinematics to GSPs are composed of five steps.

(1) Generate the GSP constraint graph according the external constraint added between the base and the platform.

(2) Add internal constraints according the initial relative position of the primitives in the platform to ensure the total degrees of freedom of primitive vertices are equal to the total degrees of constraint vertices.

(3) Well-constrained completion the GSP and generate the directed bipartite graph.

(4) If there exists a saturated vertex, where each of the constraint imposed to the vertex is external constraint, find the construction sequences.

(5) If there is no saturated vertex, to whom each of the constraint imposed is external constraint, check the external constraint types. If the total DOCs of angular constraints are three, impose these angular constraints to obtain the rotation matrix firstly and impose the rest distance constraints to obtain the translation matrix secondly.

With above five steps, parts of the problem of direct kinematics to GSPs can be transformed into polynomial equation systems with smaller size. For these smaller polynomial equation systems, we can use Wu-Ritt characteristic method [4] and complete discrimination system for polynomials [5] to obtain their real solutions and parameter conditions of the real solutions.
Experiments

Let $\pi_1$ and $\pi_2$ be the planes in the base ($\pi_1$ is on $xOy$ coordinate plane of the global coordination system) and the platform ($\pi_2$ is on $xOy$ coordinate plane of the local coordination system), respectively. $\mathbf{u}_1 = (0,0,1)$ is the normal vector of plane $\pi_1$ and $\mathbf{u}_2 = (0,0,1)$ is the normal vector of plane $\pi_2$. Let points $B_1(0,0,0)$ and $B_2(b,0,0)(b \neq 0)$ be on plane $\pi_1$ and points $P'_1(0,0,0)$ and $P'_2(a,0,0)(a \neq 0)$ be on plane $\pi_2$.

Let $R = (r_{ij})_{3 \times 3}$ be rotation matrix (i.e. $R^T = R^T R^T = I$), the determinant of $R$ is 1 and is denoted by $\det(R) = 1$ and $T = (t_1, t_2, t_3)^T$ be translation matrix. The global coordination of points $P'_1(0,0,0)$ and $P'_2(a,0,0)$ are $P_1 = (t_1, t_2, t_3)$ and $P_2 = (ar_1 + t_1, ar_2t_1 + t_2, ar_3 + t_3)$ respectively.

The external constraints added between the base and the platform are $\text{DIS}(\pi_1, \pi_2) = d_1$ (planes $\pi_1$ and $\pi_2$ are parallel, i.e. $\mathbf{u}_1$ is parallel to $\mathbf{u}_2$), $\text{DIS}(B_1, P_1) = d_2$, $\text{DIS}(B_2, P_1) = d_3$ and $\text{DIS}(B_2, P_2) = d_4$.

Its type is $4D2A$ and the corresponding equation system is as follows.

\begin{align*}
& t_{31} = 0 \\
& r_{13} = 0 \\
& |t_3| = d_1 \\
& t_1^2 + t_2^2 + t_3^2 = d_1^2 \\
& (t_1 - b)^2 + t_2^2 + t_3^2 = d_2^2 \\
& (ar_1 + t_1 - b)^2 + (ar_2t_1 + t_2)^2 + (ar_3 + t_3)^2 = d_3^2 \\
& R^T R = R^T R^T = I \\
& \det(R) = 1
\end{align*}

(1) We need to solve the first five equations of equations system (1) involving $\{t_{23}, r_{13}, t_3, t_2, t_1\}$ to fix $\pi_2$ and $P_1$. The corresponding characteristic set obtained by Wu-Ritt characteristic set method is as follows, where $r_{23} = 0$ and $r_{13} = 0$.

\begin{align*}
& t_3 \pm d_1 = 0 \\
& 4b^2t_2^2 - 2b^2(d_1^2 + d_2^2) + b^2(b^2 + 4d_1^2) + (d_2^2 - d_3^2)^2 = 0 \\
& 2bt_1 - b^2 - d_2^2 + d_3^2 = 0
\end{align*}

The number of real solutions is determined by the coefficients of the second equation of characteristic set (2). There are four real solutions at most, if $4b^2(d_1^2 + d_2^2) \geq b^2(b^2 + 4d_1^2) + (d_2^2 - d_3^2)^2$.

(2) Point $P_2$ can be fixed by the external constraint $\text{DIS}(B_2, P_2) = d_4$ and internal constraints $\text{DIS}(P_1, P_2) = a$ and $\text{DIS}(\pi_2, P_2) = 0$.

If $t_1 \neq b$, the corresponding characteristic set obtained by Wu-Ritt characteristic set method is

\begin{align*}
& 2a(t_1 - b)r_{11} + 2at_2r_{21} + c = 0 \\
& 4a^2[(b - t_1)^2 + t_2^2]r_{21} + 4at_2cr_{21} + c^2 - 4a^2(t_1 - b)^2 = 0 \\
& r_{31} = 0
\end{align*}
where \(c = a^2 + (t_1 - b)^2 - d_i^2 + t_2^2 + t_3^2\). The number of real solutions is determined by the coefficients of the second equation of characteristic set (3).

If \(t_1 = b\), and \(t_2 \neq 0\) the corresponding characteristic set obtained by Wu-Ritt characteristic set method is

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\begin{align*}
4a^2t_2r_{11}^2 + [(a - t_2)^2 + t_3^2 - d_i^2][(a + t_2)^2 + t_3^2 - d_i^2] &= 0 \\
2at_2r_{21} + a^2 - d_i^2 + t_2^2 + t_3^2 &= 0 \\
2t_1 &= 0
\end{align*}
\]

The number of real solutions is determined by the coefficients of the first equation of characteristic set (4).

(3) We need to solve the rest equations of equation system (1), i.e. \(RR^T = R^T R^T = I\) and \(\det(R) = 1\) to obtain the real solutions to \(\{r_{33}, r_{12}, r_{22}, r_{12}\}\). There are two characteristic sets and the solutions are shown as \(\{r_{33} = 1, r_{12} = 0, r_{22} = r_{11}, r_{22} = -r_{21}\}\) and \(\{r_{33} = -1, r_{12} = 0, r_{22} = r_{11}, r_{22} = r_{21}\}\).

The conditions of parameters \(\{a, b, d_1, d_2, d_3, d_4\}\) can be confirmed by operating above three steps, according to the characteristic sets and complete discrimination system for polynomials.[5]

For example, the parameter conditions to real solution

\(\{r_{11} = 0, r_{12} = 1, r_{22} = 0, r_{23} = 0, r_{33} = 0, r_{32} = 0, r_{32} = -1, t_3 = 2, t_2 = 0, t_1 = 0\}\)

are \(d_1 = 2, d_2 = 2, d_3 = \sqrt{b^2 + 4}, d_4 = \sqrt{a^2 + b^2 + 4}\).

Conclusions

We present a simple and efficient method based on graph theory and symbolic computation method. Firstly, the problem of direct kinematics to a GSP is transformed into a constraint graph. Secondly, the constraint graph is well-constrained completed according to the external and internal constraints added. Thirdly, the constraint graph is decomposed into smaller ones if possible. Finally, the smaller problems are solved with Wu-Ritt characteristic method and complete discrimination system for polynomials. With this method, we can not only decompose the GSPs into smaller ones and find the GSPs to be fixed one by one, but also obtain the corresponding parameters conditions of real solutions to direct kinematics to the GSPs.

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References


