Stability of Singularly Perturbed Stochastic Systems with Discrete and Distributed Delays

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Abstract. The problems of stability for a class of the singularly perturbed stochastic systems with discrete and distributed delays are studied. By using a Lyapunov–Krasovskii functional approach and terms of linear matrix inequalities (LMIs), stability in mean square criteria ensuring asymptotically stability of the singularly perturbed stochastic systems with discrete and distributed delays are established. New asymptotically stable in mean square criteria for the singularly perturbed stochastic systems with time-varying delay are obtained as well. Finally, the validity of the obtained results is shown by a numerical example.

Introduction

Singularly perturbed systems often occur naturally in many branches of applied mathematics and practical engineering, such as fluid dynamics, elasticity, chemical reactor theory, flight, missile, see [1], [2] and the references therein. Since time delay exists commonly in dynamic systems and is frequently a source of instability, oscillation, and poor performance, much work has been done about time- delay differential systems. Singular parameter and stochastic perturbations are both unavoidable in many physical systems. During the past decades, the robust stability and control problem for singularly perturbed stochastic delayed systems has been extensively investigated in [3–13]. For example, robustness of exponential stability of stochastic differential delay equations is first considered in [7]. Delay- dependent stability for a class of stochastic systems with time delay and nonlinear uncertainties is studied in [13]. The problem of delay-dependent stability in the mean square sense for stochastic systems with time-varying delays, Markovian switching, and nonlinearities are also investigated in [12]. Then the exponential stability in mean square for stochastic systems with multiple delays is investigated in [3]. Very recently, the problems of delay-dependent robust stability for singularly perturbed stochastic systems by introducing some free weighting matrices, which can be selected properly to lead to much less conservative results [11]. In spite of these results being nice and effective, the stability condition is still conservative.

Singularly perturbed stochastic systems with discrete and distributed delays have also received much attention in recent years since it has a wider application into aircraft stabilization, chemical engineering processes, distributed networks, neural networks, nuclear reactors and population dynamics, etc. This paper presents some simple stability criteria for singularly perturbed stochastic systems with discrete and distributed delays and for singularly perturbed stochastic systems with time-varying delay. For simplicity, we discuss only the case of a delay, which are constant but unknown. We will establish, based on the Lyapunov–Krasovskii functional approach, a new asymptotically stability in mean square by
terms of linear matrix inequalities (LMIs). This criteria can be theoretically proved to be less conservative. Numerical example shows that the results obtained in this note are effective and are an improvement over existing criteria.

Preliminaries

In this section, we introduce some preliminary concepts which will be found useful in the paper. Consider the following singularly perturbed stochastic systems with discrete and distributed delays

\[
x(t) = \phi(t), t \in [-h, 0],
\]

where

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad E(\varepsilon) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon, I_{n_2} \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix},
\]

and the positive scalar \( \varepsilon, \varepsilon_a, \varepsilon_b \) is a small singular perturbed parameter, and \( \varepsilon \) is their upper bound. \( x(t) \in \mathbb{R}^n \) is the state vector with \( x_1(t) \in \mathbb{R}^{n_1} \) and \( x_2(t) \in \mathbb{R}^{n_2}, \quad n_1 + n_2 = n \). The matrices \( A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times q}, \quad C \in \mathbb{R}^{p \times n} \) are real constant matrix with satisfy \( A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times q}, \quad C \in \mathbb{R}^{p \times n} \), and \( A(\varepsilon) = E(\varepsilon)A, \quad B(\varepsilon) = E(\varepsilon)B, \quad C(\varepsilon) = E(\varepsilon)C \). \( \omega(t) \) is a Brownian motion defined on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with a natural filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). Let \( g(t, x(t), x(t-h(t))) \) is locally Lipschitz continuous and satisfies the linear growth condition as well, and satisfy

\[
g(t,0,0) = 0,
\]

\[
\text{trace}[g^T(t, x(t), x(t-h(t)))Pg(t, x(t), x(t-h(t)))) \leq x(t)^T D_1 x(t) + x(t-h(t))^T D_2 x(t-h(t))],
\]

\( D_1 \) and \( D_2 \) are positive-definite matrices with appropriate dimensions. \( \phi(t) \) is a smooth vector-valued initial function; \( h(t) \) is a time-varying delay in the state, and \( h \) is an upper bound on the delay \( h(t) \).

Notations: Let \( \mathbb{R}^n \) be Euclidean space with vector norm \( \| \cdot \| \), the notation \( P > 0 \) for \( (n \times n) \)-matrix \( P \) means that \( P \) is symmetric and positive definite. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions. \( \mathbb{E} \) denotes the expectation operator with respect to some probability measure \( \mathbb{P} \). Throughout this paper, the following notation will be used. The symmetric terms in a symmetric matrix are denoted by * , e.g.,

\[
\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}.
\]

The following definition and lemma will be used in establishing our main results.

Definition 1 The trivial solutions of the stochastic systems (1) is said to be stochastically asymptotically stable in mean square if the following condition holds

\[
\lim_{t \to \infty} \mathbb{E}\|x(t)\|^2 = 0, \forall t > 0.
\]
Lemma 1 [14] Set
\[
Z(\varepsilon) = \begin{bmatrix}
Z_1 + \varepsilon Z_3 & \varepsilon Z_5^T \\
Z_5 & Z_2 + \varepsilon Z_4
\end{bmatrix}.
\]

If there exist matrices \(Z_i (i = 1, 2, \cdots, 5)\) with \(Z_i = Z_i^T (i = 1, 2, 3, 4)\) satisfying
\[
Z_i > 0 \begin{bmatrix}
Z_1 + \varepsilon_0 Z_3 & \varepsilon_0 Z_5^T \\
Z_5 & Z_2 + \varepsilon_0 Z_4
\end{bmatrix} > 0,
\]
then
\[
E(\varepsilon)Z(\varepsilon) = Z(\varepsilon)^T E(\varepsilon) > 0, \forall \varepsilon \in (0, \varepsilon_0).
\] (3)

Lemma 2 [15] For any constant symmetric matrix \(M \in \mathbb{R}^{n \times n}, M > 0\), scalar \(\gamma > 0\), vector function \(\psi : [0, \gamma] \rightarrow \mathbb{R}^n\) such that the integrations concerned are well defined, then
\[
-\gamma \int_{-\tau}^{0} \psi(t + s)^T M \psi(t + s) ds \leq -\left[ \int_{-\gamma}^{0} \psi(t + s) ds \right]^T M \left[ \int_{-\gamma}^{0} \psi(t + s) ds \right].
\] (4)

Main Results

In this section, this paper finds new stability criteria less conservative than the existing results. For singularly perturbed stochastic systems (1) with discrete and distributed delays, we shall give asymptotically stable in mean square condition in terms of linear matrix inequalities (LMIs) by the Lyapunov–Krasovskii functional approach, without using any model transformations and bounding techniques.

The result on the asymptotical stability in mean square of singularly perturbed stochastic systems (1) with discrete and distributed delays is presented in the following theorem.

Theorem 1 Let \(h(t)\) be a differentiable function, satisfying \(h'(t) \leq \mu < 1, t \geq 0\),

the singularly perturbed stochastic systems (1) with discrete and distributed delays is asymptotically stable in mean square if there exist matrices with appropriate dimensions \(P_i (i = 1, 2, \cdots, 5)\) and \(P_i = P_i^T (i = 1, 2, 3, 4)\), \(Q_i > 0\), \(Q_i \geq 0, Z_i \geq 0\) \((i = 1, 2)\), and for any scalar \(\varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)\) such that
\[
P_i > 0 \begin{bmatrix}
P_1 + \varepsilon_1 P_3 & \varepsilon_3 P_5 & \varepsilon P_4 \\
P_5 & P_2 + \varepsilon_0 P_4
\end{bmatrix} > 0, \begin{bmatrix}
P_1 + \varepsilon_1 P_3 & \varepsilon_3 P_5 & \varepsilon P_4 \\
\varepsilon_3 P_5 & \varepsilon P_4
\end{bmatrix} > 0,
\] (5)

and
\[
\Xi = \begin{bmatrix}
\Xi_{11} & 0 & 0 & 0 \\
0 & \Xi_{22} & \Xi_{23} & 0 \\
0 & \Xi_{32} & \Xi_{33} & 0 \\
0 & 0 & 0 & \Xi_{44}
\end{bmatrix}
\] (6)

where
\[
\begin{align*}
\mathcal{Z}_{11} &= P_0^T A(\varepsilon_a) + \varepsilon_a^T \bar{P}_0^T A(\varepsilon_a) + A(\varepsilon_a)^T P_0 + \varepsilon_a A(\varepsilon_a)^T \bar{P}_0 + D_i + Q_i + Q_2 - Z_i - \frac{1 - \mu}{h} Z_2, \\
\mathcal{Z}_{12} &= P_0^T B(\varepsilon_a) + \varepsilon_a^T \bar{P}_0^T B(\varepsilon_a) + Z_i + \frac{1 - \mu}{h} Z_2, \\
\mathcal{Z}_{13} &= -A(\varepsilon_a)^T P_0 C - \varepsilon_a A(\varepsilon_a)^T \bar{P}_0 C, \\
\mathcal{Z}_{22} &= D_i - (1 - \mu) Q_i - Z_i - \frac{1 - \mu}{h} Z_2, \\
\mathcal{Z}_{23} &= -B(\varepsilon_a)^T P_0 C - \varepsilon_a B(\varepsilon_a)^T \bar{P}_0 C, \\
\mathcal{Z}_{33} &= -Q_1, \\
\mathcal{Z}_{34} &= Q_3, \\
\mathcal{Z}_{55} &= -Q_3, \quad P_i = \begin{bmatrix} P_i \ 0 \\ P_i \end{bmatrix}, \quad \bar{P}_0 = \begin{bmatrix} P_i \ P_i^T \\ 0 \ P_i \end{bmatrix}. 
\end{align*}
\]

Now, we consider the case in which the system has no uncertainties, i.e., the singular perturbed parameter satisfies \( \varepsilon_a = \varepsilon_a = \varepsilon_b = 1 \). Consider the following certain stochastic systems with discrete and distributed delays.

\[
\begin{align*}
\frac{dx(t)}{dt} &= [Ax(t) + Bx(t - h(t)) + Cx(t - h(t))]dt + g(t, x(t), x(t - h(t)))d\omega(t), t \geq 0, \\
x(t) &= \phi(t), t \in [-h, 0]. 
\end{align*}
\]

For this case, the following theorem holds.

**Corollary 1** Let \( h(t) \) be not differentiable or the upper bound of the derivative of \( h(t) \) is unknown, the certain stochastic systems (7) with discrete and distributed delays is asymptotically stable in mean square if there exist matrices with appropriate dimensions \( P > 0, Q_i \geq 0, Q_2 > 0, Z \geq 0 \), such that

\[
\Psi = \begin{bmatrix}
\psi_{11} & \psi_{12} & \psi_{13} & 0 & 0 \\
* & \psi_{22} & \psi_{23} & 0 & 0 \\
* & * & \psi_{33} & 0 & 0 \\
* & * & * & \psi_{44} & 0 \\
* & * & * & * & \psi_{55}
\end{bmatrix} < 0,
\]

where

\[
\begin{align*}
\psi_{11} &= PA + A^T P + D_i - Z_i, \\
\psi_{12} &= PB, \\
\psi_{13} &= -A^T P C + Z_i, \\
\psi_{22} &= D_i, \\
\psi_{23} &= -B^T P C, \\
\psi_{33} &= -Q_1 - Z_i, \\
\psi_{44} &= Q_2 + hZ, \\
\psi_{55} &= -Q_3. 
\end{align*}
\]

When \( C = 0 \), time-varying delay system in (1) reduces to

\[
\begin{align*}
E(\varepsilon_x) \frac{dx(t)}{dt} &= [A(\varepsilon_a) x(t) + B(\varepsilon_a) x(t - h(t))]dt + g(t, x(t), x(t - h(t)))d\omega(t), t \geq 0, \\
x(t) &= \phi(t), t \in [-h, 0]. 
\end{align*}
\]

In this case, by Theorems 1, it is easy to derive the following result.

**Corollary 2** Let \( h(t) \) be a differentiable function, satisfying

\[
h'(t) \leq \mu < 1, t \geq 0,
\]

the singularly perturbed stochastic systems (8) with time-varying delay is asymptotically stable in mean square if there exist matrices with appropriate dimensions \( P_i (i = 1, 2, \ldots, 5) \) and \( P = P_i^T (i = 1, 2, 3, 4) \), \( Q_i \geq 0 \), \( Q_2 > 0 \), \( Z_i \geq 0 \) \((i = 1, 2)\), and for any scalar \( \varepsilon_a, \varepsilon_b, \varepsilon_c \in (0, \varepsilon_0) \) such that
$$P_i > 0, \begin{bmatrix} P_i + \varepsilon_i P_3 & \varepsilon_i P_5 \\ P_3 & P_2 + \varepsilon_i P_4 \end{bmatrix} > 0, \begin{bmatrix} P_i + \varepsilon_i P_3 & \varepsilon_i P_5 \\ \varepsilon_i P_5 & \varepsilon_i P_2 + \varepsilon_i^2 P_4 \end{bmatrix} > 0,$$

and

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 & 0 \\ * & \Omega_{22} & 0 & 0 \\ * & * & \Omega_{33} & 0 \\ * & * & * & \Omega_{33} \end{bmatrix} < 0.$$  

where

$$\Omega_{11} = P(\varepsilon_i)^T A(\varepsilon_i) + A(\varepsilon_i)^T P(\varepsilon_i) + D_i + Q_i - Z_i + \frac{\mu - 1}{h} Z_2, \Omega_{12} = P(\varepsilon_i)^T B(\varepsilon_i) + Z_i - \frac{\mu - 1}{h} Z_2,$$

$$P(\varepsilon_i) = (P_0 + \varepsilon_i^{\hat{P}})^{-1},$$

$$\Omega_{22} = D_2 - (1 - \mu)Q_i - Z_i + \frac{\mu - 1}{h} Z_2, \Xi_{33} = Q_3 + h(Z_i + Z_2) = -Q_2, P_0 = \begin{bmatrix} P_1 & 0 \\ P_3 & P_2 \end{bmatrix}, \hat{\Pi}_0 = \begin{bmatrix} P_3 & P_4 \\ 0 & P_4 \end{bmatrix}.$$  

**Numerical Example**

In this section, we will present a numerical example to illustrate the usefulness of our main results. Consider the singularly perturbed stochastic systems (9) with discrete and distributed delays, data of which are as follows

$$E = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}, C = \begin{bmatrix} -0.0558 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 1.7073 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -2.5026 & -1.0540 \\ 1 & 0.5 \end{bmatrix},$$

$$g(t, x(t), y(t)) = \sqrt{0.05} \text{diag} \{ x_1 \sin(x_1 y_1) + x_2 \cos(x_2 y_2), y_1 \cos(x_1 y_1) + y_2 \sin(x_2 y_2) \},$$

with initial state $[-1, 1]^T$, $h = 3$ and $\mu = 0$, $\varepsilon = 0.1$, $\varepsilon_a = \varepsilon_b = 0.9$.

It is easy to calculate that

$$D_i = D_2 = \sqrt{0.1}I,$$

and we verify that all the conditions of Theorem 1 are satisfied. Thus, the singularly perturbed stochastic system (9) with discrete and distributed delays is asymptotically stable in mean square, and its state chart is shown in Figure.1 below. It is clear that the state of system (9) reaches equilibrium point after 340 second.

When $C_a = 0$, we verify that all the conditions of Corollary 2 are satisfied. Thus, the singularly perturbed stochastic system (9) with time-varying delay is asymptotically stable in mean square, and its state chart is shown in Figure.2 below. It is clear that the state of system (9) reaches equilibrium point after 160 second.
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Conclusion

This note presents some new stability criteria for singularly perturbed stochastic systems with discrete and distributed delays. New techniques were developed to make the criteria less conservative. The asymptotically stable in mean square condition is obtained by using a Lyapunov–Krasovskii functional approach and terms of linear matrix inequalities (LMIs). It is noted that this condition is obtained without using any model transformations and bounding techniques. A numerical example demonstrates the validity of these methods. The results show that the methods described in this note are very effective.

References


