Algorithm for the Numerical Solution of the Inverse Problem of Filtration Theory

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Abstract. This paper deals with the algorithm for the numerical solution of the inverse problem of determining the permeability of the porous medium equation for unsteady filtration of a homogeneous liquid in an elastic inhomogeneous porous medium. The idea of using the method of modulating functions for the solution of inverse problems goes back to J. Loeb and D. Kahena.

Introduction

In mathematical physics are usually considered various direct problems: given the differential equation and the initial and (or) the boundary conditions to be satisfied by the solution of a differential equation. Setting each of the direct problems assumes a certain number of job functions. Some of these functions defined by differential equations, for example, the coefficients of the equation, the other part defined by the boundary conditions. As a result of the direct problem solution given set of functions is associated with a new feature - the direct problem solution. Let some of those function which are usually set in the direct problem are unknown, though, their finding is great interest, but instead given some additional information about solution of the direct problem. Such tasks are called inverse problems of mathematical physics. Additional information about solution of the direct problem (or solution of a series of direct problems) can be set in a different form. This may be the solution given on some set or the integral characteristics of the solution.

One of the important tasks in the oil and gas sector is the problem of determining the parameters of the stratum of the observed values of pressure, saturation, and others with monitoring wells. The main characteristics of oil reservoir are filtration and capacitive parameters. The idea of using the method of modulating functions for the solution of inverse problems goes back to J. Loeb and D. Kahen [1, 2].

The main methods of determining the parameters of systems (such as regression analysis, variation methods, deterministic methods of moments and others.) are described by differential equations, based on the solutions of equations. The weakness of the existing methods for determining the filtration parameters should include the following states:

- Low accuracy defined parameters;
- The difficulty of processing the experimental data;
- The impossibility of interpreting difficult cases filtration movement;
- A large number of design models and formulas;
- Formulas are difficult and no universal, etc.

The reason for all the shortcomings is the use solutions of direct problems.

The basic idea of modulating functions (M - method) is that the filtration equations are replaced with an integral analog that make up the algebraic or integral equations for the unknown parameters. Note that the resulting integral expressions are not derivatives of experimental functions. And as you know, the operation of differentiation of experimental functions is incorrect.
**Formulation of the Problem**

Consider the equation of unsteady filtration elastic homogeneous fluid in an inhomogeneous elastic porous medium [3]:

\[
\frac{\partial}{\partial x} \left( k(x, y) \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( k(x, y) \frac{\partial p}{\partial y} \right) = \beta' \mu \frac{\partial p}{\partial t},
\]

\((x, y) \in Q \subset \mathbb{R}^2, t \in (0, T)\)  \hspace{1cm} (1)

where \(p(x, y)\) -the reservoir pressure,

\(k(x, y)\) -the permeability of the porous medium,

\(m\) -the porosity of the medium,

\(\mu\) -the absolute viscosity of the fluid,

\(\beta'_m\) -elastic coefficient of volume expansion of the liquid,

\(\beta'_c\) -coefficient of compressibility of the porous medium,

\[
\beta' = \beta_m + \frac{\beta'_c}{m},
\]

with initial and boundary conditions

\[
p(x, y, 0) = p_0(x, y), (x, y) \in Q,
\]

\[
Lp|_{\partial\Omega} = p_1(x, y, t), (x, y) \in \Gamma, t \in [0, T]
\]

where \(L\) -the operator corresponding to the boundary conditions (for example, \(L\) -the identity operator in the case of the first boundary problem; in the case of the second boundary value problem \(L = \frac{\partial}{\partial n}\) -the derivative along the outer normal to the boundary of the region, etc.)

The inverse problem is to find the coefficient \(k(x, y)\) in the equation (1) with known functions \(m, \mu, \beta'\). We expand the function \(k(x, y)\) by the Maclaurin formula

\[
k(x, y) = k_0 + k_1 \cdot x + k_2 \cdot y + k_3 \cdot xy + k_4 \cdot x^2 + k_5 \cdot y^2 + ...
\]

(5)

After substituting (5) into (1) we have

\[
\frac{\partial}{\partial x} \left[ k_0 p'_{x} + k_1 \cdot x \cdot p'_{x} + k_2 \cdot y \cdot p'_{x} + k_3 \cdot xy \cdot p'_{x} + k_4 \cdot x^2 \cdot p'_{x} + ... \right]
\]

\[
+ \frac{\partial}{\partial y} \left[ k_0 \cdot p'_{y} + k_1 \cdot x \cdot p'_{y} + k_2 \cdot y \cdot p'_{y} + k_3 \cdot xy \cdot p'_{y} + k_4 \cdot x^2 \cdot p'_{y} + k_5 \cdot y^2 \cdot p'_{y} + ... \right] = \beta' \mu \frac{\partial p}{\partial t}
\]

Multiply both sides of (6) by smooth functions \(\varphi_i(x), \varphi_j(y), \varphi_i(t)\), \(i = 0, 1, ..., n\) (where \(\varphi_i(x), \varphi_j(y)\) -function of class \(C^2\), \(\varphi_i(t)\) -function of class \(C^1\) ) and integrate respectively

\[x_0 \leq x \leq x_1, y_0 \leq y \leq y_1, t_0 \leq t \leq t_1.\] For our case \(t_0 = 0\). Then
\[
\int_{t_0}^{t_1} \int_{y_0}^{y_1} \varphi_i(t) \varphi_i(y) dy dt \left[ k_0 \cdot \int_{x_0}^{x_1} p_{xx}'' \varphi_i(x) dx + k_1 \cdot \int_{x_0}^{x_1} (x \cdot p_i'')' x \cdot \varphi_i(x) dx + k_2 \cdot \int_{x_0}^{x_1} y p_{xx}^2 \cdot \varphi_i(x) dx + \ldots \right] + \\
+ \int_{t_0}^{t_1} \int_{y_0}^{y_1} \varphi_i(t) \cdot \varphi_i(x) dx dt \left[ k_0 \cdot \int_{y_0}^{y_1} p_{yy}'' \varphi_i(y) dy + k_1 \cdot \int_{y_0}^{y_1} x \cdot p_{yy}'' \varphi_i(y) dy + k_2 \cdot \int_{y_0}^{y_1} (y \cdot p_i'')' y \cdot \varphi_i(y) dy + \ldots \right] = \\
= \beta' m \int_{t_0}^{t_1} \int_{y_0}^{y_1} \varphi_i(y) \cdot \varphi_i(x) dx dy \int_{t_0}^{t_1} p_i' \cdot \varphi_i(t) dt, \ i = 0, n
\]

Choosing a modulating functions \( \varphi_i(x), \varphi_i(y), \varphi_i(t) \), satisfying the conditions.

\( \varphi_i(x_0) = \varphi_i(x_1) = 0, \varphi_i(y_0) = \varphi_i(y_1) = 0, \varphi_i(t_0) = \varphi_i(t_1) = 0, \varphi_i'(x_0) = \varphi_i'(x_1) = 0, \varphi_i'(y_0) = \varphi_i'(y_1) = 0, i = 0, n \)

and applying the formula for integration by parts twice in the integrals in which the present \( P_{xx}^*, P_{yy}^* \) and once to integrals, in which there are \( P_x', P_y', P_t' \), we obtain

\[
\int_{t_0}^{t_1} \int_{y_0}^{y_1} \varphi_i(t) \varphi_i(y) dy dt \left[ k_0 \cdot \int_{x_0}^{x_1} p \cdot \varphi_i''(x) dx + k_1 \cdot \int_{x_0}^{x_1} p(x \varphi_i' \cdot (x))' dx + k_2 \cdot \int_{x_0}^{x_1} y p \cdot \varphi_i''(x) dx + \ldots \right] + \\
+ \int_{t_0}^{t_1} \int_{y_0}^{y_1} \varphi_i(t) \cdot \varphi_i(x) dx dt \left[ k_0 \cdot \int_{y_0}^{y_1} p \varphi_i''(y) dy + k_1 \cdot \int_{y_0}^{y_1} x p \varphi_i''(y) dy + k_2 \cdot \int_{y_0}^{y_1} p(y \cdot \varphi_i' (y))' dy + \ldots \right] = \\
= -\beta' m \int_{t_0}^{t_1} \int_{y_0}^{y_1} p \varphi_i(x) \varphi_i(y) \varphi_i'(t) dx dy dt, \ i = 0, n
\]

The system of equations (7) can be rewritten as:

\[
k_0 \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} p \left[ \varphi_i''(x) \varphi_i(y) + \varphi_i''(y) \cdot \varphi_i(x) \right] \varphi_i'(t) dx dy dt + \\
+ k_1 \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} p \left[ (x \varphi_i')' \varphi_i(y) + x \varphi_i''(y) \varphi_i(x) \right] \varphi_i'(t) dx dy dt + \\
+ k_2 \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} p \left[ (y \cdot \varphi_i'(y))' \varphi_i(x) + y \cdot \varphi_i''(x) \cdot \varphi_i(y) \right] \varphi_i'(t) dx dy dt + \\
+ \ldots = -\beta' m \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} p \varphi_i(x) \varphi_i(y) \varphi_i'(t) dx dy dt, \ i = 0, n
\]

To determine the coefficient \( k(x, y) \) we need to find \( k_0, k_1, \ldots, k_n, \) from the system of linear algebraic equations (8).

As the modulating functions you can use the functions \( \varphi(x) = \alpha(x)(x-x_0)(x_1-x) \) in the equations of the form where there are derivatives of 1-order and \( \varphi(x) = \alpha(x)(x-x_0)^2(x_1-x)^2 \) in the equations with derivatives of 2-order, where \( \alpha(x) \) - some weighting function. The factor \( \alpha(x) \) provides the necessary form of modulation functions to \([x_0, x_1]\).
The Algorithm for the Numerical Solution of the Inverse Problem of Filtration Theory by Modulating Functions

In the expansion (5) considering the first \((n + 1)\) summands, we write a system of linear algebraic equations (8) in a matrix form

\[
A \cdot K = B
\]

where elements of the matrix \(A = (a_{ij})\) are known integrals of the unknowns \(k_0, k_1, \ldots, k_n\), for example,

\[
a_{i0} = \int_{y_0}^{y_1} \int_{x_0}^{x_1} p \left[ \varphi^n(x) \varphi(y) + \varphi^n(y) \varphi(x) \right] \varphi_i(t) dx dy dt,
\]

\[
a_{i1} = \int_{y_0}^{y_1} \int_{x_0}^{x_1} p \left[ x \varphi'(x) \varphi(y) + x \varphi'(y) \varphi(x) \right] \varphi_i(t) dx dy dt,
\]

\[
a_{i2} = \int_{y_0}^{y_1} \int_{x_0}^{x_1} p \left[ y \varphi'(y) \varphi(x) + y \varphi'(x) \varphi(y) \right] \varphi_i(t) dx dy dt, \quad i = \overline{0, n}
\]

(10)

\[
B = (b_0, b_1, \ldots, b_n)^T, \quad b_i = -\beta' \mu \int_{x_0}^{x_1} \int_{y_0}^{y_1} P \varphi_i(x) \varphi(y) \varphi'(t) dx dy dt, \quad i = \overline{0, n}
\]

(11)

\(K = (k_0, k_1, \ldots, k_n)^T\) is vector of unknowns \(k_i, \ i = \overline{0, n}\).

To uniquely determination of the \((n + 1)\) unknown \(k_i, \ i = \overline{0, n}\), we have \((n + 1)\) equations, and the determinant of the linear algebraic equations (9) must not be equal to zero.

For this select the modulating functions as follows

\[
\varphi_i(x) = x^i (x - x_0)^2 (x_1 - x)^2, \quad x \in [x_0, x_1],
\]

\[
\varphi_i(y) = y^i (y - y_0)^2 (y_1 - y)^2, \quad y \in [y_0, y_1],
\]

\[
\varphi_i(t) = t^i (t - t_0) (t_1 - t), \quad t \in [t_0, t_1], \quad i = \overline{0, n}
\]

(12)

From the equations (9)-(12) it is clear that the question of the numerical realization of the problem comes down to the question of the numerical integration of triple integrals. Then we obtain the following algorithm for the numerical solution of the inverse problem of filtration theory.

1. Reception and processing of input data. To solve the problem you first need to have experimental data on \(p(x, y, t)\) values at the nodes of a cubic lattice. In the absence of data in all the lattice points can be interpolated value of \(p(x, y, t)\).

2. Selection of the modulating functions \(\varphi_i(x), \varphi_i(y), \varphi_i(t)\), satisfying conditions (12). It is convenient to present modulating functions as follows:

\[
\varphi_j(x) = x^j \sum_{i=0}^{4} \omega^x_i x^i, \quad \varphi_j(y) = y^j \sum_{i=0}^{4} \omega^y_i y^i, \quad \varphi_j(t) = t^j \sum_{i=0}^{2} \omega^t_i t^i, \quad j = 0, 1, \ldots, n,
\]

where \(\omega^x_i, \omega^y_i, \omega^t_i\) respectively:
\[ \omega_i^+ = x_i x_2, \quad \omega_i^- = -2(x_1 + x_2)x_i x_2, \]
\[ \omega_2^+ = (x_1 + x_2)^2 + 2x_1 x_2, \quad \omega_3^+ = -2(x_1 + x_2), \quad \omega_4^+ = 1, \]
\[ \omega_2^- = y_i y_2, \quad \omega_i^- = -2(y_1 + y_2)y_i y_2, \]
\[ \omega_i^+ = (y_1 + y_2)^2 + 2y_1 y_2, \quad \omega_i^- = -2(y_1 + y_2), \quad \omega_i^- = 1, \]
\[ \omega_i^- = t_i t_2, \quad \omega_i^- = (t_i + t_2), \quad \omega_i^- = -1. \]

3. Filling in the values of the elements of the matrix A and B (9), namely the values of \( a_{ij} \) and \( b_i \)
(10)-(11), which are triple integrals. Calculation of triple integrals in (10)-(11) reduces to the
calculation of repeated integrals using known formulas of numerical integration (Newton-Cotes
formulas, etc.).

4. Finding cond (A)-the condition number of the matrix A:

\[ \text{cond}(A) = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}, \]

where \( \lambda_{\text{max}} \) - the maximum and \( \lambda_{\text{min}} \) - the minimum eigenvalues of A. To overcome the difficulties
associated with incorrect of ill-conditioned systems, you must ensure the "closeness" of the
condition number of A to 1. This is achieved by varying modulating functions or averaging original
data.

5. The solution of linear algebraic equations (9).

Suitably selecting modulating function, you can ensure that the conditions of non-degeneracy of
the matrix A: \( \det A \neq 0. \)

To solve the system (9) can be used, for example, the well-known Gaussian elimination with
pivoting. As a result, the solution of (9) we find the vector of unknowns \( K = (k_0, k_1, \ldots, k_n)^T. \)

6. Substituting the obtained values of \( k_i, \quad i = 0, n \) in the expansion (5) of the required functions
\( k(x, y). \)

Conclusion

1. M-method allows you to move from incorrect differentiation experimental function to correct
operation of integration.

2. In the algorithms for solving inverse problems using the M-method is not the main source of
error-use solutions of the direct boundary value problems.

3. A practical source of errors can be numerical methods for solving integrals and approximation
of the experimental functions.

4. One of the main advantages of this method is its simplicity and efficiency of practical
application.

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