Analysis of Stochastic Time-dependent Transportation Network
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**Keywords:** stochastic time-dependent transportation network; nash equilibrium; intelligent transportation systems.

**Abstract.** Modeling transportation network as a stochastic and time-dependent network has become a trend. Instead of exploring the dynamic of the stochastic and time-dependent properties of transportation networks, a majority of the literature treat the statistical distribution of the transportation network as a fixed function and try to find an optimal path or an optimal routing policy. In this paper, we analyze the transportation network from the perspective of game theory and try to find the driven force that sharpens the stochastic and time-dependent properties of transportation network. We model the travel time as a random process that is determined not only by the length of the corresponding link, but also by the number of drivers that are traveling on this link. Based on this modeling method, we model the path selection process of all the drivers in the network as a non-cooperative game, where all the drivers try to maximize their utility function by selecting a path that is emanating from their points of departure to their destinations. A day-to-day repeated game process is addressed and a formal definition of Nash equilibrium is also given. At last, the main contributions of this paper is summarized and the future work is discussed.

**Introduction**

Intelligent transportation systems (ITS) have became a hot research area for many years. Currently, it is fashionable and generally accepted that we should model transportation network as a stochastic and time-dependent (STD) graph, which means the travel time of a certain link of the network is modeled as a random variable, whose probability density function (PDF) is time-varying. Unlike the traditional modeling method, where there is a lot of highly efficient shortest path finding algorithms can be used, such as Dijkstra algorithms [1], A-star algorithms [2] and so on [3,4,5], there doesn't exist highly efficient shortest path finding algorithms that can be applied to a STD network. So far, a great deal of scholars' effort is being devoted to the invention of highly efficient shortest path finding algorithms in STD network [6,7,8,9,10,11,12]. We also addressed the problem of route guidance in STD network in another paper by using multi-lognormal distribution to characterize the statistical property of transportation network [13].

However, through our literature review, we found that most of the literature implicitly assume that the statistical property of the network is independent from the behavior of the drivers in it and no matter which path the decision maker choose, it will exert no influence on the general statistical condition of the whole network. That is to say, this assumption totally neglects the competitive and interactive effect during drivers' decision making process.

Presumably, all the drivers in the network are inclined to choose a path that is more faster and more reliable, which means the selected path should have less expected travel time and the variance of travel time is relatively small. Considering this scenario: in figure 1, there are two divers A and B in this network and they are at the same place, which is node 1, heading for the same destination,
which is node 2. The expected travel time of path 1 and path 2 are 8+X1 minutes and 3+X2 minutes, where X1,X2 denote the number of drivers on path 1 and path 2, respectively. Driver A departs at 9 o'clock and driver B departs at 5 minutes past 9 o'clock. It's self-evident driver A would choose the path that is optimal at 9 o'clock, which is path 2. Needless to say, as a result, the network's statistical property will be changed, due to the action of driver A, the expected travel time of path 1 will increase to 13 minutes, therefore, driver B, instead of making the same decision like driver A, will choose path 1. This example reveals the competitive and interactive effect when the drivers are making decisions about route selection.

![Path Diagram](image)

Figure 1. A simple example.

In this paper, we try to reveal the driven force that sharpens the stochastic and time-dependent properties of transportation network from the perspective of game theory. The remaining part of this paper is organized as follow: In section II, for the purpose of convenient presentation, we give some basic definitions of transportation network and develop some notations. Notably, we try to model the travel time of a certain road as a random process determined by the number of motor vehicles on the road. In section III, to analyze the stochastic and time-dependent properties of transportation network, we try to establish a framework by utilizing the intellectual fruit of game theory. It is worth mentioning that when a driver decide to take some path, the expected travel time is not only dependent on the attributes of the transportation network, but also dependent on the decisions other drivers make. In section IV, under this framework developed in section III, we try to analyze the route selection process of drivers though day-to-day repeated game process and a formal definition of equilibrium is given. In section V, the main contributions of this paper is summarized and the future work is also discussed.

**Formulation of Network Characteristics**

In this section, we develop some basic definitions and try to model the competitive and interactive effect between drivers of motor vehicles in the same transportation network when they are making decisions of path selection.

*Some Basic Definitions.* It's self-evident that a transportation network can be abstracted to a directed graph $G=(N,L,D,C)$. Without loss of generality, we assume that the graph has a static topology and the practical meaning of the link cost is the travel time of the corresponding link. The definitions of the symbols are given below:

- $N$ is the set of nodes and the number of the nodes in the networks is $|N|=n$, then $N=\{d_1,d_2,\ldots,d_n\}$.
- $L$ is the set of directed links and $l_{i,j}$ denotes the directed link emanating from node $d_i$ to node $d_j$. Therefore, we have $L=\{l_{i,j} \mid d_i,d_j \in N\}$ and the number of links in graph is $|A|=m$.
- $D$ is the set of lengths and $d_{i,j}$ denotes the length of link $l_{i,j}$, then we have $D=\{d_{i,j} \mid l_{i,j} \in L\}$. 

C is the set of link costs and $c_{i,j}(t)$ denotes the cost of link $l_{i,j}$ when the arriving time at node $i$ is time $t$, then we have $C = \{c_{i,j}(t) | \forall l_{i,j} \in L \}$.

**Modeling of Travel Time Tables.** It is self-evident that the travel time of a certain road is dependent on the number of drivers that are traveling on this road. If there are more drivers on $l_{i,j}$ at time $t_1$ than at time $t_2$, $c_{i,j}(t_1)$ should be much bigger than $c_{i,j}(t_2)$. In this subsection, we try to mathematically formulize this idea.

Obviously, more vehicles on a road, lower the traveling speed of all the vehicles. Let $v_{i,j}(t)$ denotes the instantaneous velocity of motor vehicles traveling on link $l_{i,j}$ at time $t$, then $v_{i,j}(t)$ should be watched as a real-valued function of $x(t)$, where $x(t)$ denotes the number of motor vehicles that are traveling on link $l_{i,j}$ at time $t$. The mathematical presentation is given below:

$$v_{i,j}(t) = f_{i,j}(x(t)). \quad (1)$$

where $f_{i,j}(\cdot)$ is a non-increasing real-valued function corresponding to link $l_{i,j}$.

Figure 2.A shows an example of the number of vehicles at different time $t$ denoted as $x(t)$. Figure 2.B shows a typical form of $f_{i,j}(\cdot)$. When the number of vehicles is relatively small, in this case, that is smaller than 30, the traveling speed is maximized. When the number of vehicles is beyond some threshold, which is 30 in figure 2.B, the traveling speed will radically drop.

Without consideration of stochastic property, for any given time $t$, through the definition of (1), $c_{i,j}(t)$ should be the solution of the equation of definite integration below:

$$\int_t^{t + c_{i,j}(t)} v_{i,j}(u)du = \int_t^{t + c_{i,j}(t)} f_{i,j}(x(u))du = d_{i,j}. \quad (2)$$

Since time $t$ is given, $f_{i,j}(\cdot)$ can be obtained through statistical analysis of traffic data and $d_{i,j}$ can be accessed to very easily, once we get $x(t)$, $c_{i,j}(t)$ can be obtained through solving the equation above.
According to equation (1), the instantaneous velocity can be calculated and is depicted in figure 3.A. According to equation (2), if the length of $l_{i,j}$ is the 400 meters, the traveling time on $l_{i,j}$ when the entering time is $t$ denoted as $c_{i,j}(t)$ can also be calculated and is depicted in figure 3.B.

So far, we have added the time-varying ingredient to the transportation network, now we need to add the stochastic ingredient. State the obvious fact: more vehicles on the road, higher the probability that the traffic congestion will happen on the road. Now we give the formal and final definition of $c_{i,j}(t)$ below:

$$c_{i,j}(t) = \bar{c}_{i,j}(t) + g_{i,j}(x(t)) \cdot \alpha_{i,j}(t). \quad (3)$$

where $\bar{c}_{i,j}(t)$ is the solution to equation (2), $g_{i,j}()$ is also a non-decreasing real-valued function corresponding to link $l_{i,j}$ and $\alpha_{i,j}(t)$ is a white gaussian random process.

In summary, we model link cost $c_{i,j}(t)$ as a random process, whose statistical property is determined not only by $d$, but also by the behavior of all the drivers in the transportation network, which sharpens $x(t)$. We believe through this modeling method, the competitive and interactive effect of drivers when they are making decisions about path selection can be taken into consideration.

**Mathematical Framework**

From the discussion in section II, we have illustrated that $c_{i,j}(t)$ is not only determined by $d_{i,j}$, which is a build-in attribute of the transportation network, but also determined by $x(t)$, which is sharpened by the behavior of the drivers in the network. In this section, through application of non-cooperative game theory, we analyze the path selection process of drivers in the network.

According to Nash [14], a n-person game should consist of three basic elements: game players, strategy sets and utility functions. Therefore, in the first place, we should define the counterparts of the three elements in a transportation network.

**Game Players.** The game players should be all the motor vehicle drivers in the transportation network. Suppose there are $n$ drivers that can be considered as the users of the transportation network and let $I$ be the set of players, then we have $I=\{1,2,3...n\}$.

For every player $i$ in $I$, there is a 3-tuple $\{s_{i}, d_{i}, t_{i}\}$ associated with player $i$, where $s_{i}$, $d_{i}$, $t_{i}$ denotes start node, destination node and start time, respectively.

**Tables.** **Strategy Sets.** A strategy of player $i$ should represent an optional path that can be selected by player $i$. For player $i$, the associated strategy set is $S_{i} = \{r_{1}^{i}, r_{2}^{i}, r_{3}^{i}\}$.

Every element $r_{j}^{i}$ in $S_{i}$ should be a path emanating from $s_{i}$ to $d_{i}$. It is worth mentioning that since the number of nodes and the number of links are both finite, therefore, the strategy set of any player of this game is a finite set, which means this game is a finite game.

**Utility Functions.**

The utility function should capture the idea that every driver will make the decision of path selection leading to less travel time and more reliability. Therefore, the utility function of player $i$ denoted as $U_{i}$ should have this form below:

$$U_{i} = U_{i}(E, D). \quad (4)$$
where E and D are expectation and variance of the travel time of the selected path, respectively.

Without loss of generality, suppose the player i chooses $r_k^i$ as his strategy, to calculate his utility, according to equation (4), we should calculate the expectation and variance of travel time of path $r_k^i$. However, given the discussion in section II, since we modeled each link cost as a random process, to calculate the expected travel time of path $r_k^i$, we should find the statistical distribution of all the link costs in C, which is dependent on the behavior of all the drivers in the transportation network. That is to say, the statistical distribution of the whole transportation network is dependent on the strategies selected by all the players in this game. Figure 5 illustrates this point.

![Figure 4. A typical form of function $g_{ij}(\cdot)$ and $c_{ij}(\cdot)$ obtained through equation (3).](image1)

![Figure 5. Utility of any player is determined by the strategies of all the players.](image2)

In summary, the utility function $p_i$ of player i should maps the n-tuple $\{r_1^i, r^2_i, r^3_i, \ldots\}$ into the real numbers, where $\{r_1^i, r^2_i, r^3_i, \ldots\}$ denotes a collection of strategies of all the players 1,2,3,...,n. We give the formal definition of utility function $p_i$ as below:

$$U_i = U_i(r_1^i, r^2_i, r^3_i, \ldots).$$  \hspace{1cm} (5)

**Day-to-Day Repeated Game Process**

Based on the framework established above, we model the path selection process as a day-to-day repeated game, which means all the players will play this game every day. We believe this modeling method is close to reality, due to the fact that a majority of the drivers in a realistic transportation network drive for commuting and their home, working places and working time do not change very often.
Besides, we take the assumption that every player in this game have access to the history statistical condition of the whole network. We think this assumption is also close to reality, since every player can observe the history data of the number of vehicles in the network at any time t, from which the statistical distribution can be derived based our modeling method in section II.

Now we analyze the behavior of every player in this game. For every player i in I, let \( r_i(t) \) be the selected strategy on day t, where t belongs to \( \{0,1,2,3\ldots\} \). Without loss of generality, based on our assumption, when player i wants to make the decision of path selection on day t, the collection of strategies \( \{r_i(t-1), r_i^2(t-1), r_i^3(t-1), \ldots, r_i^n(t-1)\} \) is revealed to him. We believe that player i will choose the optimal path calculated through the statistical distribution on day t-1. Therefore, for every player i in I, \( r_i(t) \) should satisfy the condition below:

\[
U_i(r_i^1(t-1), \ldots r_i^n(t-1)) \geq U_i(r_i^1(t-1), \ldots r_j^1(t-1), \ldots r_i^n(t-1)), \quad \forall r_j^1 \in S_i . \tag{6}
\]

According to the equation (6), if on some day t, for every player i in I, the equation \( r_i(t-1)=r_i(t) \) is true, then for any day after t, every player's strategy will not change any more. In game theory, this is the so called nash equilibrium point [14]. It is obvious that this game is a finite game, therefore, there exist at least one nash equilibrium point in this game.

**Summary**

In this paper, we try to analyze the stochastic and time-dependent properties of transportation network from the perspective of game theory. Unlike the traditional method, which tries to find a statistical distribution to characterize the transportation network, we modeled the link cost, whose practical meaning is the travel time of the corresponding link, as a random process determined not only by the length of the link, but also by the number of vehicles on this road. Through this modeling method, we established a framework based on game theory, where the competitive and interactive effect when drivers are making decisions of path selection can be aptly incorporated into. A day-to-day repeated game process is presented and a definition of Nash equilibrium in this game is also given. In the future, we try to refine this model based on analysis of realistic traffic data. Through the analysis of realistic traffic data, we also want to check if drivers' behavior is consistent with the proposed day-to-day repeated game process.

**References**


