Optimum Plan for Constant-Stress Accelerated Life Test with Censoring and Numerical Simulation

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Abstract. In this paper, we obtain the optimum plan by discussing a constant-stress accelerated life test (ALT) satisfying some specific condition at k stresses under an exponential distribution.

Introduction


Kou Hai-xia and An Zong-wen [18] studied double synchronous-step-down-stress accelerated life testing. This paper gives the optimum plan for a constant-stress ALT which satisfies the condition \[ \sum r = r \] under an exponential distribution at k≥3 constant stresses by minimizing the asymptotic variance of MLE in line with a linear accelerating equation.

Basic Assumption and Lemmas

A constant-stress ALT with censoring can be designed as following:

We sample \( n(\geq r) \) products at random, supposing the unit number at stress level \( s_i \) to be \( n_i \) (i=1,...,k), divide them into k groups and test the life times of the products at stress level \( s_i \) (\( s_1 < s_2 < s_3 < \ldots < s_k \)) respectively till \( r_i \) products fail, where \( \sum n_i = n, \sum r_i = r, s_i (s_i < s_k) \) is a usual stress level, the failure data at stress level \( s_i \) are denoted by \( t_{i1}, \ldots, t_{ik} \) respectively (i=1,...,k). Let the life times of the products satisfy the following assumptions:

A1: The life times of the products at stress \( s_i \) follow an exponential distribution with \( F_i(t) = 1 - e^{-\lambda_i t}, t \geq 0, i = 1, \ldots, k; \theta_i = 1/\lambda_i \) is a mean life at stress \( s_i \);

A2: The accelerating equation between mean life \( \theta \) and stress \( s \) is: \( \ln \theta = a + b\phi(s) \), where \( \phi(s) \) is a decreasing function of \( s \).

According to the test data and the basic assumptions, the likelihood function is

\[
L = \left( \prod_{i=1}^{k} \lambda_i^{r_i} \right) \exp\left( -\sum_{i=1}^{k} \lambda_i T_i \right)
\]

(2.1)
where $T_i = \sum_{j=1}^{r_i} t_{ij} + (n_i - r_i)t_{ir_i}$ is the total time of the test at stress $S_i$ ($i=1,\ldots,k$).

By (2.1) we have:

$$\ln L = \sum_{i=1}^{k} r_i \ln \lambda_i - \sum_{i=1}^{k} \lambda_i T_i = -\sum_{i=1}^{k} r_i \ln \theta_i - \sum_{i=1}^{k} T_i / \theta_i$$  \hfill (2.2)

From A2, we have $\theta_i = e^{a_i + b_i \Phi_i}$, and put it into (2.2), then

$$\ln L(a, b) = -\sum_{i=1}^{k} r_i (a + b \Phi_i) - \sum_{i=1}^{k} T_i e^{-(a + b \Phi_i)}$$  \hfill (2.3)

Lemma 2.1: When testing with censoring II, we have $E(T_i) = \gamma_i \theta_i$, $i = 1,\ldots,k$.

Proof: There are $T_i \sim \Gamma (r_i, 1/\theta_i)$, $i = 1,\ldots,k$, when testing with censoring II, then $E(T_i) = \gamma_i \theta_i$, $i = 1,\ldots,k$.

Asymptotic Variance of the Log Mean

To compute the Fisher information matrix of the log likelihood function (2.3)

$$E\left( \frac{\partial^2 \ln L(a, b)}{\partial a^2} \right) = E\left( \sum_{i=1}^{k} T_i e^{-(a + b \Phi_i)} \right) = \sum_{i=1}^{k} e^{-\lambda_i} \lambda_i T_i E(T_i) = \sum_{i=1}^{k} e^{-\lambda_i} \lambda_i T_i e^{-(a + b \Phi_i)} = \sum_{i=1}^{k} r_i \Phi_i = A_{11}$$  \hfill (3.1)

$$E\left( \frac{\partial^2 \ln L(a, b)}{\partial b^2} \right) = E\left( \sum_{i=1}^{k} e^{-(a + b \Phi_i)} \Phi_i T_i \right) = \sum_{i=1}^{k} e^{-\lambda_i} \lambda_i T_i E(T_i) = \sum_{i=1}^{k} e^{-\lambda_i} \lambda_i T_i e^{-(a + b \Phi_i)} = \sum_{i=1}^{k} r_i \Phi_i = A_{11}$$  \hfill (3.2)

$$E\left( \frac{\partial^2 \ln L(a, b)}{\partial a \partial b} \right) = E\left( \sum_{i=1}^{k} e^{-(a + b \Phi_i)} \Phi_i \right) = \sum_{i=1}^{k} e^{-\lambda_i} \lambda_i T_i E(T_i) = \sum_{i=1}^{k} e^{-\lambda_i} \lambda_i T_i e^{-(a + b \Phi_i)} = \sum_{i=1}^{k} r_i \Phi_i = A_{11}$$  \hfill (3.3)

Then the Fisher information matrix of the likelihood function is: $F = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}$, the inverse matrix of F is:

$$F^{-1} = \frac{1}{\Delta} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{12} & A_{11} \end{pmatrix}$$  \hfill (3.4)

And so the asymptotic variance of the log mean life at usual stress is:

$$\text{AsVar}(\ln \theta_i) = (1, \Phi_0) F^{-1} (1, \Phi_0)' = 1/\Delta (A_{22} - 2A_{12} \Phi_0 + \Phi_0 A_{11}) = 1/\Delta (\sum_{j=1}^{k} r_j (\Phi_0 - \Phi_j)^2 + \sum_{j=1}^{k} r_j (\Phi_2 - \Phi_j)^2 + \cdots + r_k r_k (\Phi_{k-1} - \Phi_k)^2)$$  \hfill (3.5)

Optimal Distribution of the Failure Number in a Constant-Stress ALT with Censoring II

To make computing and application easy, we take $k$ accelerating stress levels satisfying:

$$\left\{ \begin{array}{l} \Phi_0 - \Phi_1 = d \Delta \\
\Phi_{i-1} - \Phi_i = \Delta 
\end{array} \right.$$

where $d$ is an integer, $\Delta$ is a constant, $i=2,\ldots,k$ \hfill (4.1)

Put (4.1) into (3.5), then:

$$\text{AsVar}(\ln \theta_i) = \frac{\sum_{i=1}^{k} r_i (d + i - 1)^2}{\sum_{i=1}^{k} r_i (j - i)^2} = \frac{R}{S}$$  \hfill (4.2)
The optimum test plan in a constant-stress ALT, on one hand, requires the minimum asymptotic variance of estimator at normal stress so as to improve the preciseness of statistical analysis. On the other hand, needs to satisfy that the failure numbers at higher stress levels are not smaller than that at the minimum stress level so as to get more failure data in a shorten time, moreover, the following conditions should also be satisfied:

\[
\begin{align*}
& r_i \leq r, i = 2, \ldots, k \\
& \sum_{i=1}^{k} r_i = r
\end{align*}
\]  

(4.3)

In the rest of the paper, we assume that (4.3) holds and give the optimal distribution plan of \( r \) failure numbers at \( k \) accelerating stress-levels by regarding the minimum asymptotic variance of the log mean as principle.

**Theorem 4.1** For a constant-stress test with censoring II at \( k \) stress levels, if (4.3) holds, the optimum failure numbers of transformation are: \( r_i = r/k, r/k \) is integer, \( i = 1, \ldots, k \).

**Proof:** as the proof of theorem 4.1 in [15].

**Optimal Distribution of the Sample Number in a Constant-Stress ALT with Censoring II**

One of the purposes to accelerate life tests is to shorten the testing time and cut down the testing cost. Therefore, one can assume that the mean times of the constant-stress ALTS at \( k \) stress levels are equal, i.e.

\[
E(t_{1n}) = E(t_{2n}) = \ldots = E(t_{kn})
\]

(5.1)

From [14] \( (P_{120}) \), we know that:

\[
E(t_i) = \frac{1}{n} \sum_{j=0}^{n-i-1} \frac{n}{n_i + 1} = \frac{1}{n} \left( \frac{1}{n_i - 1} + \frac{1}{n_i - 2} + \ldots + \frac{1}{n_i - n_i + 1} \right) = \frac{n}{\sum_{j=1}^{n-1} 1} n_i \sum_{j=1}^{n-1} 1
\]

\[
= \theta \{ \ln(n_i + c + \varepsilon_{n_i}) - \ln(n_i - r_i) \} = \theta \ln(n_i / n_i - r_i), \ i = 1, \ldots, k
\]

(5.2)

We also know \( \sum_{j=1}^{n} 1 = \ln n + c + \varepsilon \), \( c = 0.577216 \) is the Euler constant, \( \varepsilon \to 0 \) \( (n \to \infty) \).

We can get from (5.1), (5.2):

\[
\theta_i / \theta_1 = \ln(n_i / n_i - r_i) / \ln(n_i / n_i - r_i), \ i = 2, \ldots, k.
\]

(5.3)

From A2, \( \theta_1 = \exp(a + b\Phi_i) \), \( i = 1, \ldots, k \), then we have:

\[
\theta_i / \theta_1 = \exp(a + b\Phi_i) / \exp(a + b\Phi_i) = \exp[b(\Phi_i - \Phi_i)], \ i = 2, \ldots, k
\]

(5.4)

From (5.3), (5.4) We get:

\[
\ln(n_i / n_i - r_i) / \ln(n_i / n_i - r_i) = \exp[b(\Phi_i - \Phi_i)], i = 2, \ldots, k
\]

Furthermore,

\[
\left( \ln(n_i / n_i - r_i) \right)^{1/(\theta_i - \theta_1)} / \left( \ln(n_i / n_i - r_i) \right)^{1/\theta_1} = e^{\theta}, \ i = 2, \ldots, k
\]

(5.5)

From (5.5), We can get:

\[
\left( \ln(n_i / n_i - r_i) / \ln(n_i / n_i - r_i) \right)^{1/(\theta_i - \theta_1)} = \left( \ln(n_i / n_i - r_i) / \ln(n_i / n_i - r_i) \right)^{1/\theta_1} \ i = 2, \ldots, k
\]

(5.6)

and

\[
\ln(n_i / n_i - r_i) = \left( \ln(n_i / n_i - r_i) \right)^{1/(\theta_i - \theta_1)} / \left( \ln(n_i / n_i - r_i) \right)^{1/\theta_1} \ i = 2, \ldots, k
\]

(5.7)
If both sides of (5.7) are divided by \( \ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right) \) respectively and

\( \ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right) = \left( \ln \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)^{-1} \), put into the right side of (5.7), then

\[
\frac{\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)}{\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)} = \left( \ln \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)^{-1} \left[ \left( \ln \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)^{-1} \right] = \ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)/\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)
\]

moreover,

\[
\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)/\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right) = \ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)/\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right), i = 2, \ldots, k
\]  

(5.8)

According to the power series expansion of Log function, for \( x > 0 \),

\[
\ln x = 2\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{2n+1}(\frac{x-1}{x+1})^{2n+1} \ldots 1 , \text{ And when } \ln x \approx 2\left(\frac{x-1}{x+1}\right)^3 (5.8) \text{ is changed to}
\]

\[
\frac{\left(\frac{n_{i+1}}{n_{i+1} - r_{i+1}}\right)^{2n+1} + \frac{1}{2n+1}(\frac{n_{i+1}}{n_{i+1} - r_{i+1}})^{2n+1} \ldots 1}{\left(\frac{n_{i+1}}{n_{i+1} - r_{i+1}}\right)^{2n+1} + \frac{1}{2n+1}(\frac{n_{i+1}}{n_{i+1} - r_{i+1}})^{2n+1} \ldots 1}, i = 3, \ldots, k.
\]

which can be simplified to:

\[
\frac{r_{i+1}(2n_{i+1} - r_{i+1})}{r_{i+1}(2n_{i+1} - r_{i+1})} = \frac{r_{i+1}(2n_{i+1} - r_{i+1})}{r_{i+1}(2n_{i+1} - r_{i+1})}, \quad i = 3, \ldots, k
\]  

(5.9)

When \( r_{i+1} = r_{i+1} = \cdots = r_{i+1} = r/k \) (By theorem 4.1 the asymptotic variance of the log mean is minimum), one can get from (5.9):

\[
\frac{2n_{i+1} - r/k}{2n_{i+1} - r/k} = \frac{2n_{i+1} - r/k}{2n_{i+1} - r/k} \ldots \frac{2n_{i+1} - r/k}{2n_{i+1} - r/k}
\]  

(5.10)

implies the sequence:

\[
2n_{i+1} - r/k, 2n_{i+1} - r/k, \ldots 2n_{i+1} - r/k
\]  

(5.11)

is a geometric sequence, and \( \sum_{i=1}^{k} n_{i+1} - n \) is satisfied. Let the common ratio be \( q \), then the sum of the former \( k \) items is \( \frac{2n_{i+1} - r/k}{1-q^k} \), and it is also shown with \( \sum_{i=1}^{k} (2n_{i+1} - r/k) = 2\sum_{i=1}^{k} n_{i+1} - n \cdot r/k = 2n - r \).

From the two formulas above, we have:

\[
\frac{2n_{i+1} - r/k}{2n_{i+1} - r/k} = \frac{1-q^k}{1-q^k}
\]  

(5.12)

To sum up, we can get:

**Theorem 5.1** The sequence \( 2n_{i+1} - r/k, 2n_{i+1} - r/k, \ldots 2n_{i+1} - r/k \) is a geometric sequence with a common ratio \( q \) and \( \frac{2n_{i+1} - r/k}{2n_{i+1} - r/k} = \frac{1-q^k}{1-q^k} \).

**Proof:** Easy to be obtained through (5.10) and (5.12).

Theorem 5.2 The common ratio is \( q = e^{-b_1} \) to the geometric sequence \( 2n_{i+1} - r/k, \ldots 2n_{k} - r/k \).

Proof: From (5.3) and (5.4), the following formula is obtained:

\[
\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)/\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right) = \exp \left( b(\Phi_1 - \Phi_i) \right), \quad i = 2, \ldots, k
\]

(*)

and from (4.1) one can get: \( \Phi_1 - \Phi_i = (i-1)\Delta \), \( i = 2, \ldots, k \), and put it into (*), then,

\[
\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)/\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right) = \exp \left( b(i-1)\Delta \right), \quad i = 2, \ldots, k
\]

(5.13)

and

\[
\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right)/\ln \left( \frac{n_{i+1}}{n_{i+1} - r_{i+1}} \right) = \exp \left( b(i-2)\Delta \right), \quad i = 2, \ldots, k
\]

(5.14)
(5.13) is divided by (5.14), then

$$\frac{\ln\left(\frac{n_i}{n_i-r_i}\right)}{\ln\left(\frac{n_{i+1}}{n_{i+1}-r_{i+1}}\right)} = e^{\alpha i} \quad i = 2, \ldots, k$$

(5.15)

And it is gotten that the common ratio is $q = e^{\alpha}$ to the geometric series $2n_i - r/k, 2n_i - r/k, \ldots 2n_i - r/k$ through (5.8) (5.9) (5.10) (5.11) and (5.15).

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**References**


