Isogeometric Contact Analysis Treated with Mortar-based Method

Yi-shen ZHANG, Guo-lai YANG, Hu-tian FENG and Qing-si CHENG
Nanjing University of Science & Technology, Nanjing, Province Jiangsu, P.R. China

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Abstract. An augmented method as well as mortar-based approach for Isogeometric contact problem was presented in this paper. The contact constrains were examined through a projection to control point quantities, which can effectively improve the smoothness in global contact forces. To alleviate the influence of the normal contact penalty parameter, an augmented treatment was applied to the Isogeometric contact analysis, which can provide essentially exact satisfaction of constraints with finite penalties, and decrease the ill-conditioning of governing equations. The given numerical example and results can demonstrate the robustness and accuracy of mortar-based treatment and the utility of augmented method.

Introduction

Isogeometric analysis can be regarded as a successful merging of computer-aided geometry design (CAGD) and the finite element method (FEM) [1]. It has been shown that Isogeometric analysis provides many advantages over standard finite element analysis (FEA). With using NURBS as basis functions, it can exactly describe some common shapes and achieve geometric precision even at the coarsest mesh. The NURBS mesh can be refined or change the order of continuity without communicating to the CAD model which opens a pathway for integration of CAGD and analysis processes [2]. The Isogeometric analysis might improve the modeling of contact problems [3]. In order to alleviate these problems, various geometrical smoothing techniques have been developed based on Hermite interpolation [4,5], B’ezier [5,6] and NURBS descriptions [7]. These procedures generally improve the performance of the contact algorithms by enhancing the continuity of the contact surfaces [1]. However they do not increase the order of convergence since the higher order approximations involve only the surface but not the bulk behavior of the solids. A systematic mortar-based study of contact problems with Isogeometric analysis was initiated by Temizer et al. [3] using NURBS discretization. De Lorenzis et al. [1], subsequently observed that higher-order NURBS discretization can deliver global algorithmic smoothing effects and ensure the local quality of the solution in a two-dimensional mortar-based approach with friction.

Mostly, the contact problem was treated with the penalty approach, which provide obvious advantages: the technique is quite simple, and is readily interpreted from a physical standpoint [8]. This formulation introduces no additional equations or field variables, but it is also well-known that ill-conditioning is inevitable and become even worse when penalty values are increased. Constraints are satisfied exactly only in the limit of infinite penalty values [9]. The method of augmented Lagrangians, originally proposed by Hestenes [10] and Powell [11] in the context of mathematical programming problems subject to equality constraints, has been known to provide important advantages over the traditional Lagrange multiplier and penalty methods.

In this paper, we apply the augmented method on Isogeometric contact analysis as well as mortar-based treatment. The formulation of contact analysis using augmented method will be presented. For simplicity, the friction law considered will be a Coulomb law, with no distinction made between static and kinematic coefficients of friction. The mortar-based NURBS discretization was illustrated. Numerical example was presented and the results were compared and discussed.
Isogeometric Contact Analysis with Augmented Treatment

Contact Discretization with Augmented Treatment

Generally, we assume that two bodies in contact are elastic and all the external forces are potential forces. Let \( c = \gamma^{(1)} \cap \gamma^{(2)} \) denote the contact surface. The total potential is written as:

\[
\Pi = \Pi_e + \Pi_{ext} + \Pi_c
\]  

(1)

Where \( \Pi_e \) is the elastic energy, \( \Pi_{ext} \) is the potential of external forces, and \( \Pi_c \) is the contact potential. Here \( p \) is the Lagrangian multiplier, and \( g_N \) is the gap function. They are subjected to the Kuhn–Tucker condition. The penalty regularization replaces the multiplier with the penalty functional:

\[
\Pi_c = \frac{1}{2} \int \epsilon \left( -g_N \right)^2 da
\]  

(2)

Where \(<\cdot>\) stands for the Macaulay bracket. This formulation allows for some small penetration controlled by the parameter \( \epsilon \). The variation of \( \Pi_c \) gives

\[
\delta \Pi_c = \int \epsilon g_N \delta g_N da = \int \epsilon \left[ X^{(1)} - X^{(2)}(V_p) \right] \cdot \left( \delta X^{(1)} - X^{(2)} \right) da
\]  

(3)

\( X^{(1)} \) is the given point on \( \gamma^{(1)} \), which can be given on the contact surfaces

\[
X^{(1)} = \sum_{i \in Ncrl^{(1)}} N_i^{(1)}(u^1, u^2) q_i^{(1)}
\]  

(4)

\( X^{(2)}(V_p) \) is the unique closest point on \( \gamma^{(2)} \), which can be given on the contact surfaces

\[
X^{(2)} = \sum_{i \in Ncrl^{(2)}} N_i^{(2)}(v^1, v^2) q_i^{(2)}
\]  

(5)

Where \( N_i^{(j)} \) is NURBS shape function, \( q_i^j \) is control point, and \( Ncrl^{(j)} \) represents the set of control points for each body. Thus, we can get:

\[
\delta \Pi_c = -\sum_{i \in Ncrl^{(1)}} R_i^{(1)} \delta q_i^{(1)} - \sum_{i \in Ncrl^{(2)}} R_j^{(2)} \delta q_j^{(2)}
\]  

(6)

The contact residual vectors are identified:

\[
R_i^{(1)} = \sum_{l \in G^{(1)}} W_l J^{(1)}(u_l) N_i^{(1)}(u_i) \left[ X^{(2)}(V_{pi}) - X^{(1)}(u_i) \right]
\]  

(7)

\[
R_j^{(2)} = \sum_{l \in G^{(2)}} W_l J^{(2)}(u_l) N_j^{(2)}(V_{pi}) \left[ X^{(1)}(u_j) - X^{(2)}(V_{pi}) \right]
\]  

(8)

Here \( W_l \) is the integration weight, \( J^{(1)}(u_l) \) is the surface Jacobian on \( \gamma^{(1)} \), \( G^{(2)} \) is the set of Gauss points that in contact with \( \gamma^{(2)} \). To present the stiffness matrix, we introduce the notations as

\[
K = \begin{bmatrix}
K^{(11)} & K^{(12)} \\
K^{(21)} & K^{(22)}
\end{bmatrix}, \quad K^{ij} = \frac{\partial R_i}{\partial q_j}
\]  

(9)

Augmented Lagrangian Formulation

The concept of the method is remarkable simple, we formulate the contact pressure as

\[
p_N = \epsilon_N g_N + \lambda_N
\]  

(10)
The variational equation is written as

$$\Pi_e + \Pi_{ext} + \int_c (\epsilon_N g_N + \lambda_N) g_N da = 0$$

(11)

We can see that the equation (9) is a penalization of the Lagrange multiplier problem, which is exact if the multipliers are the correct ones. If $\lambda_N$ is the correct multipliers, then $g_N = 0$ on $c$. Here we regard $\lambda_N$ as a fixed current estimate of the correct Lagrange multiplier, and solve the problem

$$\Pi_e + \Pi_{ext} + \int_c <\epsilon_N g_N + \lambda_N^{(k)} > g_N da = 0$$

(12)

Where $\lambda_N^{(k)}$ denotes the fixed estimate of the correct $\lambda_N$. The superscript $(\bullet)^{(k)}$ reflects the fact that the search for the correct $\lambda_N$, is an iterative process. One notes that the term $<\epsilon_N g_N + \lambda_N^{(k)}>$ plays the role of the exact Lagrange multiplier in (11).

If we update the $\lambda$ as:

$$\lambda_N^{(k+1)} = \left\{ \lambda_N^{(k)} + \epsilon g_N^{(k)} \right\}$$

(13)

The gap $g_N^{(k)}$ will keep decreasing, and ultimately, the $\lambda_N$ will approach the contact pressure $p_N$.

The following table shows the augmented algorithm for frictionless contact.

Table 1. Augmented algorithm for frictionless contact.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Initialization: Set $\lambda_N^{(1)} = 0, k = 1$</td>
</tr>
<tr>
<td>2.</td>
<td>Solve for $g_N^{(k)}$</td>
</tr>
<tr>
<td></td>
<td>$\delta \Pi = \int_c \left( \lambda_N^{(k)} + \epsilon g_N^{(k)} \right) \delta x da$</td>
</tr>
<tr>
<td>3.</td>
<td>Check for constraint satisfaction:</td>
</tr>
<tr>
<td></td>
<td>IF($g_N^{(k)} \leq TOL$ for all area) THEN</td>
</tr>
<tr>
<td></td>
<td>Converge. EXIT</td>
</tr>
<tr>
<td></td>
<td>ELSE</td>
</tr>
<tr>
<td></td>
<td>Argument:</td>
</tr>
<tr>
<td></td>
<td>$\lambda_N^{(k+1)} = \left{ \lambda_N^{(k)} + \epsilon g_N^{(k)} \right}$</td>
</tr>
<tr>
<td></td>
<td>$k = k + 1$</td>
</tr>
<tr>
<td></td>
<td>GOTO 2</td>
</tr>
<tr>
<td></td>
<td>ENDIF</td>
</tr>
</tbody>
</table>

**Mortar-based Contact Treatment**

Starting with the discretization of the slave surface and the master surface.

$$X^{(1)} = \sum_k R_i^{(1)}(u^1, u^2)q_i^{(1)}X^{(2)} = \sum_k R_i^{(2)}(v^1, v^2)q_i^{(2)}$$

(14)

The gap of the contact surfaces $g_N = [X^{(1)} - X^{(2)}(V_p)]$ when $X^{(1)} - X^{(2)}(V_p) \leq 0$. The key ingredient of a mortar-based method is the projection of kinematic quantities to degrees of freedom.

Using the notation $\left\{ \bullet \right\} = \int_{\Omega} \bullet dA$, the projection
\[ \bar{g}_N^i \geq \langle R^i, g_N \rangle \]  \hspace{1cm} (15)

The normal pressure degrees of freedom are defined with the penalty parameter as
\[ \bar{p}_N^i = \epsilon_N \bar{g}_N^i \]  \hspace{1cm} (16)

As the normal gap \( g_N \) essentially appears in the projection integration, there is no need to associate a unique normal with each projection. To subject the Kuhn–Tucker condition, the contact constraints is satisfied by the projected quantities which determine the active set:
\[ \bar{g}_N^i \leq 0 \quad \bar{p}_N^i \geq 0 \quad \bar{g}_N^i \bar{p}_N^i = 0 \]  \hspace{1cm} (17)

The local pressure is defined via a discretization as for all other degrees of freedom on the slave surface via:
\[ p_N = \sum R^i p_N^i \]  \hspace{1cm} (18)

**Numeric Simulation and Analysis**

This example deals with the frictionless contact of two infinite long cylinders. Only a quarter of the geometry is considered, the coarse NURBS mesh and the loading conditions are summarized in Fig.1. We consider the cylinders as linearly elastic with Young’s modulus \( E = 1000 \) and Poisson’s ratio = 0.3. The radius \( R = 1 \) and the cylinder is loaded with a vertical force \( P = 10 \) applied as uniform pressure on the upper surface.

![Figure 1. The coarse NURBS mesh and the loading conditions.](image)

Four different penalty parameters are chosen to evaluate the effect of penalty parameter \( \epsilon_N \) in Fig.2 (the number of elements is 10*20). Clearly, by choosing a smaller penalty parameter \( \epsilon_N \), the maximum pressure is less than the theory value and the contact area is wider (\( \epsilon_N = 10^*E, 100^*E \)). However, the large penalty parameter \( \epsilon_N \) will cause undesirable oscillations of the contact pressure across the contact/no-contact zone as said before (\( \epsilon_N = 500^*E, 800^*E \)). As the mesh is refined (the number of elements is 20*40), the maximum pressure is closer to the theory value but the unstable pressure is still exist when choosing large \( \epsilon_N \), as seen in Fig. 3.
Figure 2. Effects of penalty parameter $\epsilon_N$ (Element number: 10*20).

Figure 3. Effects of penalty parameter $\epsilon_N$ (Element number: 20*40).
With using the mortar treatment, it is obvious to see that the unstable contact pressures caused by large $\epsilon_N$ across the contact/no-contact zone are better in Fig.4. These results highlight the robustness of the mortar approach. On the other hand, the maximum pressure is still effected by the choosing $\epsilon_N$.

![Diagram](image)

Figure 4. Effects of penalty parameter $\epsilon_N$ with mortar-based treatment(Element number: 20*40).

**Conclusions**

We have recommended an augmented treatment of isogeometric contact analysis. Numerical example has demonstrated that the method reduce the impact of the normal penalty parameter. However, the result in the current study still does not match Hertz solution, and further study is needed to improve the results. This paper focus on the frictionless contact problem, the extension to frictional contact is the task of an ongoing research.

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**References**


