High-Precision Chebyshev Spectral to Approximate Laplace Eigenvalue Problem

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Abstract. In this paper, we focused on the Laplace eigenvalue problem which solved by Chebyshev spectral based on Lagrange polynomial. The application background of this problem is mainly light film vibration, thermal magnetic radiation, and lattice vibration, etc. This paper used spectral theory to solve the Laplace eigenvalues with high-precision. Through compared we found that our method is better than the other linear finite element method, because Chebyshev spectral method is a global approximation, it has achieved superconvergence, spectral accuracy, and easier programming, we also analysis the error of this method and projecting the eigenvectors in this paper, we found that the eigenvalue vector was approximated by this method which has more stability and smoothly. At the same time, this method also provides a new and effective reference for solving Laplace eigenvalue problem.

Introduction

The Laplace eigenvalue problem is a classic research topic with extensive contents, wide application and multi-disciplines crossed. It is the basis for many eigenvalue problems researches, it is also a bridge between partial differential equations, numerical analysis and functional analysis, etc. It is also the focus of scientific research and engineering applications. Laplace eigenvalue problem has the characteristics of strong practicability, continuable extension, and relatively complete theoretical system. This problem usually applied to particle motion, thermal magnetic radiation, lattice vibration \cite{1}, light thin film vibration, graph theory \cite{2}, and Riemann manifolds theory \cite{3}.

The Laplace eigenvalue study originated from the “Kac problem”, which was put forward in 1966 by the famous American scholar Mark Kac. The problem is: “one can hear the shape of drum based on the drum’s sounds?”, Because of this he obtained the prize of The American Mathematics Chauvenet Honor Award in 1968, this issue has been pending, until 1992 it confirmed by three scholars, they are Carolyn, David Webb and Scott Wolpert \cite{4}. The answer is negative, namely, someone cannot judge the shapes of drum just according to the sounds. The point of view in their paper is that the eigenvalues of an arbitrary polygon and an octagon are equivalence, that is to say, the same eigenvalue can correspond to different eigenvalue vectors. Therefore, the uniqueness of the Kac problem cannot be guaranteed.

Laplace eigenvalue problem is an important branch of the differential operator eigenvalue theory and the basis of quantum mechanics. In addition, the movement law of microscopic particles has a close connection with it. In the 1930s, quantum mechanics came from scratch \cite{5}, and from guessing and verifying, it was finally proposed by the great scientist Schrödinger. Its great discovery has caused an influence not less than the Niudun law in classical mechanics, the wave function needs to satisfy the conditions of single value, finiteness, continuity, and normalization. The Schrödinger equation is also a simplified model for understanding free electrons in solid-state
metals, and for two-dimensional infinite deep-square wells of the equivalent form is a Laplace characteristic equation, so the Laplace eigenvalue problem is also a solution tool for solving quantum mechanics.

Laplace eigenvalue problem has been widely concerned in recent years. This method has been continuously expanded and improved. But there are also many questions about how to approach the eigenvalue problem quickly, steadily, and with high accuracy, as well as truncation of the eigenvalue solution interval (upper and lower bounds of eigenvalues). And how to better visualize eigenvectors have always been the key to the exploration of eigenvalue problems. The solution theory needs improvement. The current methods for solving Laplace eigenvalue problems is mainly finite element [6], finite difference [7], and boundary element [8], but the approximation effect is not ideal and the solution efficiency is also very general. Therefore, a better discrete method and a friendly algebraic solver need to be founded. The Chebyshev spectral distribution of discrete points is more reasonable, and the global approximation accuracy is also better. This paper uses Chebyshev spectral points combined with Lagrange interpolation polynomials to solve the Laplace eigenvalue problem. It belongs to the new extension of the spectral method, it is very worthy to research and attention.

The solution to this problem is mainly divided into two steps, the first step is discrete the differential equations, and the second step is to solve the problem of algebraic eigenvalues. Discrete focused on whether the algebraic structures of discrete matrices are symmetric, diagonally, sparse, etc. The pre-processing work will affect the accuracy, stability and convergence speed. Secondly, the dimensions of the discrete matrices should be paid attention to when solving algebraic eigenvalues. The problem to be solved is as follows. The equation is often abbreviated as $\Delta u = \lambda u$, among them $\Delta$ is a Laplace operation.

$$\begin{cases} 
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda u, & x \in (-1,1), y \in (-1,1) \\
u_{x=-1} = 0, & u_{y=1} = 0 
\end{cases}$$

Spectral methods mainly began in 1970 and were initially developed by the collection of papers by Orszag and Gottlieb [9]. The accuracy of the spectral method of solving Laplace eigenvalues is relatively ideal. Some representative scholars such as Reddy SC, who cooperated with Trefethen LN in 1990, published a pseudo-eigenvalue problem based on the Lax stability-based full discrete spectrum method [10], the book of Trefethen LN was published by SAIM in 2000, one of chapters, he converted Laplace problems into polar coordinating to solve. Besides Osting B solves the Laplace eigenvalue in 2010, his article introduces the “star domain”. The eigenvalues solved by Fourier cosine coefficients [11], and two special nonsmooth spectral functions are optimized with quasi-Newton methods, the paper also proposed eigenvalue ratio is a generalization of the Payne-Pólya-Weinberger ratio. Based on three types of special triangles (equal, isosceles right angles, and $30^\circ$, $60^\circ$, $90^\circ$ triangular), the approximation formula for the eigenvalues in any triangular domain is given, reflecting the effect of a specific geometric region on the characteristic problem. Shan W, Li H proposed a spectral method based on an arbitrary triangle to solve the Laplace eigenvalue problem in 2015 [12]. And then, Plestenjak B et al, studied the spectral collocation method to solve the multi-parameter eigenvalue problem [13], the application scope has been extended.

**Chebyshev Theory**

We describe the development of the spectral method above, next we'll introduce the Chebyshev spectral discrete process. This method is not only adaptive to solving heat conduction, wave equations, and elliptic equations, but also can solve the eigenvalue problems with high accuracy. There is a great relationship with interpolation points. The Lagrange interpolation function is selected for the interpolation function in this paper. The coefficients of the higher derivative are
highly regular, and the accuracy of the interpolation method is higher than the extrapolation method.

**Spectral Point Generation**

In constructing the spectral interpolation function, the Chebyshev point must be introduced. This is very different from the Fourier spectral method. The Fourier spectrum is mainly DFT-transformed on the equidistant grid points. Chebyshev points belong to non-isometric grid points. Its introduction can avoid the occurrence of the Long Ge phenomenon. The sequence \( x_0, x_1, x_2, ..., x_n \) is decreasing which produces by the function [14]:

\[
x_j = \cos(j\pi/N)
\]

The spectrum points of the square domain and the circular domain are shown as following. When solving the PDE problems, the square domain is generally used in the rectangular coordinate system, and the circle domain is generally in the form of polar sitting.

![Figure 1. The square domain of Chebyshev points.](image)

![Figure 2. The circle domain of Chebyshev points.](image)

**The Interpolation Theory of Lagrange Based on Chebyshev Point**

The construction of Chebyshev's interpolation function is similar to the polynomial interpolation. This paper constructs the interpolation function based on the spectral point, we chooses the Lagrange interpolation function as a base function, \( n+1 \) different Chebyshev points \( x_0, x_1, ..., x_n \) and \( f(x_0), f(x_1), ..., f(x_n) \) are one-to-one correspondence. We use interpolation polynomials \( P_n(x_i) \) to approximate the function \( f(x_i) \). The expression is as follows:

\[
P_n(x_i) = f(x_i), \quad i = 0,1,2,...
\]

The \( l_j(x) \) is a polynomial of degree \( n \) at \( n+1 \) nodes \( x_0 < x_1 < ... < x_n \), meet the conditions

\[
l_j(x_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}, \quad j,k = 0,1,...,n
\]

Those \( n+1 \) polynomials of \( l_0(x), l_1(x), ..., l_n(x) \), the base function \( l_i(x) \) specific can be expressed as a linear interpolation function.

\[
l_i(x) = \frac{(x-x_0)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)} \quad i = 0,1,...,n
\]

At this point, the interpolation function of formula (1.1) can be written the linear combination of \( l_i(x) \).

\[
P_n(x) = \sum_{i=1}^{n} l_i(x)f(x_i), \quad i = 0,1,2,...
\]

If denote the \( G_{n+1} \) as \( G_{n+1}(x) = (x-x_0)(x-x_1)...(x-x_n) \).

then \( G'_n(x_i) = (x_i-x_0)(x_i-x_1)...(x_i-x_n) \).
Therefore, the above formula (1.3) is rewritten as

\[ l_i(x) = \sum_{i=1}^{n} \frac{G_{n+i}(x)}{(x-x_i)G'_{n+i}(x_i)}, \quad i = 0, 1, 2, \ldots \]

And (1.4) can be further transformed into

\[ P_n(x) = \sum_{i=1}^{n} \frac{G_{n+i}(x)}{(x-x_i)G'_{n+i}(x_i)}, \quad i = 0, 1, 2, \ldots \quad (1.5) \]

For example, when \( N=2 \), there are three spectral point sequences \( x_0, x_1, x_2 \), \( x_0 = 1 \), \( x_1 = 0 \), \( x_2 = -1 \). Corresponding function value is \( u_0, u_1, u_2 \), the interpolation function as following.

\[ P(x) = \frac{1}{2}x(x+1)u_0 + (1+x)(1-x)u_1 + \frac{1}{2}x(x-1)u_2 \quad (1.6) \]

Take the derivative of \( P(x) \), we can gain \( P'(x) = x(x+\frac{1}{2})u_0 - 2xu_1 + (x-\frac{1}{2})u_2 \). Let’s put three spectral point coordinates \( x_0, x_1 \) and \( x_2 \) respectively put into the function \( P'(x) \). Now, it is easily to get the matrix linear equations. The coefficient matrix \( D_2 \) of under equation is called the first order spectral derivative matrix of the differential operator:

\[
\begin{pmatrix}
P'(x_0) \\
P'(x_1) \\
P'(x_2)
\end{pmatrix} = \begin{pmatrix}
\frac{3}{2} & -2 & \frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} \\
-\frac{1}{2} & 2 & -\frac{3}{2}
\end{pmatrix} \begin{pmatrix}
u_0 \\
u_1 \\
u_2
\end{pmatrix}, \quad D_2 = \begin{pmatrix}
\frac{3}{2} & -2 & \frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} \\
-\frac{1}{2} & 2 & -\frac{3}{2}
\end{pmatrix}
\]

The polynomial constructed above has a strong regularity after derivation and can be obtained by mathematical induction, \((D_N)_{ij}\) representing the elements of Chebyshev’s derivative matrix’s \( i+1 \) row and \( j+1 \) Column, and meet \( c_i = \begin{cases} 
2 & i = 0, N \\
1 & i = 1, \ldots, N-1 
\end{cases} \). The form of the Nth-order derivative matrix is

\[
D_N = \begin{bmatrix}
\frac{2N^2+1}{6} & 2(\frac{1}{1-x_j}) & \frac{(-1)^N}{2} \\
\vdots & \ddots & \frac{(-1)^{i+j}}{x_i-x_j} \\
-\frac{1}{2}\frac{(-1)^i}{1-x_j} & \frac{x_j}{2(1-x_j^2)} & \frac{1}{2}\frac{(-1)^j}{1-x_j} \\
\frac{(-1)^{i+j}}{x_i-x_j} & \vdots & \ddots \\
\frac{(-1)^N}{2} & 2(\frac{1}{1-x_j}) & \frac{2N^2+1}{6}
\end{bmatrix}
\]

Another expression is \( (D_N)_{00} = \frac{2N^2+1}{6} \), \( (D_N)_{j0} = \frac{-x_j}{2(1-x_j^2)} \), \( j = 1, \ldots, N-1 \), \( (D_N)_{jj} = \frac{c_j(-1)^{i+j}}{c_j(x_j-x_i)} \), \( i \neq j \), \( i, j = 0, \ldots, N-1 \), \( (D_N)_{NN} = -\frac{2N^2+1}{6} \).
In order to reduce the rounding errors, the above matrix $D_N$ each row of diagonal elements need to be adjusted. All elements of the row except diagonal elements should sum, and then take opposite number. At last, it can be written as $(D_N)_{ij} = \sum_{j=0}^{n} (D_N)_{ij} \cdot$

**Discrete Process**

This section we’ll introduce a Koronecker product matrix multiplication algorithm which defines a new element product method. Spectral derivation matrix is exactly the coefficient matrix of the linear equation with a strong regularity. Therefore, different orders of differential operators have the different discrete matrix and the number of spectral points will affect the dimensions of the discrete matrix. This paper selects the Koronecker product to discrete 2-dimensional Laplace operator, the results is:

$$\frac{\partial u}{\partial x^2} \to (I_N \otimes D_N^{(2)}) u_{N^2 \times 1} \quad \frac{\partial u}{\partial y^2} \to (D_N^{(2)} \otimes I_N) u_{N^2 \times 1}$$

There are two equivalent forms of the discrete matrix’s expression. Take 2D as an example. $x$ Axis and $y$ are divided into two column vectors in an equally spaced manner. $x$ axis is $x=(x_1, x_2, ..., x_N)^T$, $y$ axis is $y=(y_1, y_2, ..., y_N)^T$. All grid coordinate intersections in the 2-dimensional plane are: $(x_1, y_1), (x_1, y_2), ..., (x_i, y_1), (x_i, y_2), ..., (x_N, y_N)$. $I_N$ an identity matrix that represents an N-order matrix, the notation $\otimes$ is represente Kronecker product operation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

Generally speaking, the form of 2D Laplace operators is $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$, operation $\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial y^2}$ have the discrete form as following. $D_3^{(2)}$ often is called Chebyshev Spectrum Derivation Matrix.

$$\frac{\partial u}{\partial x^2} = (I_3 \otimes D_3^{(2)}) u_{3 \times 1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\frac{\partial u}{\partial y^2} = (D_3^{(2)} \otimes I_3) u_{3 \times 1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\frac{\partial u}{\partial x^2} = (I_3 \otimes D_3^{(2)}) u_{3 \times 1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\frac{\partial u}{\partial y^2} = (D_3^{(2)} \otimes I_3) u_{3 \times 1} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$
The second type is directly expressed in the form of matrix multiplication.

\[
\frac{\partial u}{\partial x^2} \rightarrow D_N^{(2)} u_{N\times N} \quad \frac{\partial u}{\partial y^2} \rightarrow u_{N\times N}(D_N^{(2)})^{-1}
\]

We also choose the 2-dimensional Laplace operator, the specific discrete result is:

\[
\begin{align*}
\frac{\partial u}{\partial x^2} &= D_3^{(2)} u_{3\times 3} = \\
&= \begin{bmatrix} D_{11}^{(2)} & D_{12}^{(2)} & D_{13}^{(2)} \\ D_{21}^{(2)} & D_{22}^{(2)} & D_{23}^{(2)} \\ D_{31}^{(2)} & D_{32}^{(2)} & D_{33}^{(2)} \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix} \\
\frac{\partial u}{\partial y^2} &= u_{3\times 3} D_3^{(2)} = \\
&= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} D_{14}^{(2)} & D_{12}^{(2)} & D_{13}^{(2)} \\ D_{24}^{(2)} & D_{22}^{(2)} & D_{23}^{(2)} \\ D_{34}^{(2)} & D_{32}^{(2)} & D_{33}^{(2)} \end{bmatrix}
\end{align*}
\]

**Numerical Examples**

The background of the 2-dimensional Laplace eigenvalues is the 2-D infinite deep-square well Schrodinger equation and the optical thin film vibration. The characteristics of this equation are: 1) The solution domain is a square domain and the boundary condition is the Dirichlet boundary; 2) The boundary of this equation belongs to the type of aperiodic boundary; 3) According to the current experience, the first few eigenvalues are the objects that are the focus of attention. This example mainly calculates the first five eigenvalues and compared with the finite linear element method (Literature [12]). The Laplace equation’s analytic solution is

\[
\lambda_{n_1,n_2} = \frac{\pi^2}{4} (n_1^2 + n_2^2)
\]

\[
u_{n_1,n_2} = \sin \left[ \frac{n_1 \pi}{2} (x+1) \right] \sin \left[ \frac{n_2 \pi}{2} (y+1) \right].
\]

\[
\begin{cases}
-(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u = \lambda u \\
x \in (-1,1), |y| < 1 \\
u \big|_{y=\pm 1} = 0
\end{cases}
\]

This example uses the Chebyshev spectrum to discretize the Laplace differential equation. The result of the discrete equation is \([D_N^{(2)} \otimes I_N - I_N \otimes D_N^{(2)}]u = \lambda u\). The discrete matrix is \(A = [D_N^{(2)} \otimes I_N - I_N \otimes D_N^{(2)}]\), as the same time, the discrete matrix is centrally symmetric. To deal with the boundary, we need to emphasize that we can force the boundary to be 0, otherwise eigenvalue equation will cause singularity, another method can consider SVD, but with a large error. For the small-scale discrete matrices can be directly solved by the \texttt{eig()} function that comes from MATLAB. However, a large-scale solving problem generally requires the combination of algebraic iteration method. Such as Arnoldi iterations based on Krylov subspace, Lanczos algorithm, and Jacobi-Davidson algorithm, etc. The solution of this example is small in size and can be directly use \texttt{eig()}, the eigenvector output as follows:
**Error Estimation**

Theorem 1. \( a = x_0 < x_1 < \ldots < x_n = b \), \( h = \max_{j \leq n} (x_j - x_{j-1}) \), \( f \in C^n[a, b] \). If there exists \( f^{(n+1)} \) on \([a, b]\), there is an error estimate [14].

\[
\max_{a \leq x \leq b} |R_n(x)| \leq \frac{h^{n+1}}{4(n+1)!} \max_{a \leq x \leq b} |f^{(n+1)}(x)| \quad \forall x \in [a, b]
\]

Proof: For \( \forall x \in [a, b] \), there is an integer \( k \) so that \( x \in [x_k, x_{k+1}] \), from this we can see that

\[
| (x - x_k)(x - x_{k+1}) | \leq \frac{1}{4} h^2, \quad |x - x_{k+2}| \leq 2h, \ldots |x - x_n| \leq (n-k)h,
\]

\[
| x - x_{k-1} | \leq 2h, \ldots |x - x_0| \leq (k+1)h, \quad \text{then} \quad |G_{n+1}(x)| \leq \frac{h^{n+1}}{4} \quad \text{Combining Lagrange's interpolation items},
\]

\[
R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} G_{n+1}(x) \quad \xi \in (a, b)
\]

then, it can be proved

\[
\max_{a \leq x \leq b} |R_n(x)| \leq \frac{h^{n+1}}{4(n+1)!} \max_{a \leq x \leq b} |f^{(n+1)}(x)| \quad \forall x \in [a, b]
\]

When \( N=2 \), the error estimate of the quadratic interpolation can be expressed as:

\[
| R_2(x) | \leq \frac{M_2}{3} | (x - x_0)(x - x_1)(x - x_2) |, \quad M_3 = \max_{a \leq x \leq b} |f^{(3)}(x)| \quad \forall x \in [a, b]
\]
Numerical Experimental Results

Based on the above discrete results analysis, this paper also compared the numerical results with the linear finite element. We compared error of the first five eigenvalues with the literature [12], in order to reduces the rounding error of the difference between the exact solution and the numerical solution when calculating the error. So we can simultaneously multiply $\frac{4}{\pi^2}$ to the exact solution and the numerical solution. Now, The numerical solution only need to compared with $n_1^2 + n_2^2$. In addition, the convergence order for each step is also calculated. the Chebyshev spectral points are non-equidistant, although is only the average step size when compared with the literature method, but the N is equal.

Table 1. Comparison of relative errors of Chebyshev spectrum and linear finite element approximation.

<table>
<thead>
<tr>
<th>h</th>
<th>FEM $\hat{\lambda}_1$</th>
<th>our method $\hat{\lambda}_1$</th>
<th>Rate</th>
<th>FEM $\hat{\lambda}_{2,3}$</th>
<th>our method $\hat{\lambda}_{2,3}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = \frac{1}{4}$</td>
<td>1.27e-02</td>
<td>4.62e-03</td>
<td>--</td>
<td>2.71e-02</td>
<td>2.35e-02</td>
<td>--</td>
</tr>
<tr>
<td>$h = \frac{1}{8}$</td>
<td>3.20e-03</td>
<td>4.57e-08</td>
<td>16.6</td>
<td>6.85e-03</td>
<td>3.69e-05</td>
<td>9.3</td>
</tr>
<tr>
<td>$h = \frac{1}{16}$</td>
<td>8.02e-04</td>
<td>1.25e-13</td>
<td>18.5</td>
<td>1.72e-03</td>
<td>7.11e-13</td>
<td>25.6</td>
</tr>
<tr>
<td>$h = \frac{1}{32}$</td>
<td>2.01e-04</td>
<td>1.06e-14</td>
<td>3.6</td>
<td>4.29e-04</td>
<td>1.38e-13</td>
<td>2.4</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>h</th>
<th>FEM $\hat{\lambda}_4$</th>
<th>our method $\hat{\lambda}_4$</th>
<th>Rate</th>
<th>FEM $\hat{\lambda}_5$</th>
<th>our method $\hat{\lambda}_5$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = \frac{1}{4}$</td>
<td>5.14e-02</td>
<td>9.30e-03</td>
<td>--</td>
<td>6.39e-02</td>
<td>9.30e-03</td>
<td>--</td>
</tr>
<tr>
<td>$h = \frac{1}{8}$</td>
<td>1.29e-03</td>
<td>4.62e-05</td>
<td>7.7</td>
<td>1.64e-02</td>
<td>8.58e-04</td>
<td>3.4</td>
</tr>
<tr>
<td>$h = \frac{1}{16}$</td>
<td>8.94e-04</td>
<td>8.94e-13</td>
<td>25.6</td>
<td>4.13e-03</td>
<td>1.31e-10</td>
<td>22.6</td>
</tr>
<tr>
<td>$h = \frac{1}{32}$</td>
<td>8.03e-04</td>
<td>1.29e-13</td>
<td>2.8</td>
<td>1.03e-03</td>
<td>1.12e-13</td>
<td>10.2</td>
</tr>
</tbody>
</table>

Numerical experimental results analysis: As can be seen from the above two tables, the accuracy of the Chebyshev spectral method based on Lagrange is higher than linear finite element in this paper. When N=16, the accuracy is basically reached ideal, and convergence is superconvergence. Using this method to solve 2D Lapalce eigenvalues is a very effective method. It can also be used to solve the circle domain the condition is that converted to the polar coordinates. The following figure 4 analyzes the relationship between the approximation accuracy of the first five eigenvalues and the error of the interpolation points. It can be seen from the figure that the error can eventually be reached $10^{-13}$, with the increase of the interpolation point, the accuracy of the approximation tends to be stable, and it can be basically maintained at about this amplitude. Compared with trigonal spectral elements, this method does not require triangular mesh planning [12], which can reduce the calculation amount and improve the calculation. The efficiency is basically the same from the solution accuracy and the convergence speed.

The triangulation spectral element method is spectral methods based on finite elements. First, the region is divided into triangle meshes, and then the Koornwinder-Dubiner polynomial is used to
interpolate each element. The interpolation points use spectral points. It is a local patch interpolation, so the approximation accuracy will be limited. However, due to the addition of spectral points, this method is better than traditional finite element approximations, but the accuracy and convergence speed are as well as the global approximation of Chebyshev. But the amount of programming and calculation is relatively complicated. The right figure shows the vertical projection of the feature vector. It can also be seen from the trend of the contour map. The edge of the feature vector corresponding to each feature value can be smoothly transitioned, and the basic shape of the figure can also be identified. This article only shows the 16 eigenvector diagrams, the subsequent figure approximation effects and the importance of the research will gradually decrease, so it is not necessary to present them all.

![Figure 4. The relationship between precision and interpolation points.](image)

![Figure 5. Lapalce feature vector projection.](image)

**Conclusion**

In this paper, we mainly study the Chebyshev spectral method based on Lagrange interpolation to solve Laplace eigenvalue problem. The spectrum is divided into periodic Fourier spectrum and aperiodic Chebyshev spectrum. There are advantages and disadvantages in different conditions. The Fourier spectrum has higher precision for solving periodic boundaries. Chebyshev spectral method is mainly introduced in this paper [10,15,16]. Using Chebyshev points as interpolation points, the calculation accuracy and efficiency are improved by adjusting the number of interpolation function bases and spectrum points. Through numerical experiments, we can make a conclusion: Firstly, the approximation accuracy is gradually improved with the Chebyshev point encryption. Secondly, the Lagrange interpolation spectrum method mentioned is superior to the linear finite element method. The grid step size no need very small which will converge to the ideal state. The convergence speed approximate super-convergence. At last, compared the linear finite elements, Chebyshev spectral methods no need to divide element and integrate the element, and the eigenfunctions are also well visualized. It is intuitive to compare the eigenvectors of different eigenvalues, which is beneficial to modal analysis. The study of Laplace eigenvalue can also promote the study of Schrödinger equations, singularity equations and iterative methods. At present, the higher order of spectral methods is no longer a problem, such as the two-dimensional fourth-order biharmonic problem. But it needs to be further extended to high-dimensional, combined with paralle algorithm to solve large-scale problems, making the application of spectral methods more extensive.

**References**


