Stability of Exponential Euler Method for Linear Stochastic Differential Equations with Piecewise Continuous Arguments

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Abstract. In this paper, exponential stability in mean square of the exponential Euler method to linear stochastic differential equation with piecewise continuous arguments (LSEPCAs) is showed. We give the exponential stability in mean square of the exact solution to linear stochastic differential equation with piecewise continuous arguments by using interval with integral end-points method and definition of logarithmic norm. The exponential Euler method to linear stochastic differential equation with piecewise continuous arguments is proved to share the same stability for any step size by property of logarithmic norm. Finally, an example is provided to illustrate our theories.

Introduction

Stochastic modeling has come to play an important role in many branches of science and industry. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics and finance. Most stochastic differential equations (SDEs) are nonlinear and cannot be solved explicitly, whence numerical solutions are required in practice. Numerical solutions to SDEs have been discussed under the Lipschitz condition and the linear growth condition by many authors (see[1],[2],[3],[4],). Many authors have discussed numerical solutions to stochastic delay differential equations (SDDES) (see [5], [6],[7]). The stability of the implicit Euler Scheme to SDEs is known for any step size. However, in this article we propose an explicit method to show that the exponential Euler method to stochastic differential equation with piecewise continuous arguments (LSEPCAs) is proved to share the stability for any step size by the property of logarithmic norm.

Preliminary Notation and the Exponential Euler Method

Let $B(t) = (B_1(t), \cdots, B_d(t))^T$ be a d-dimensional Brownian motion defined on the probability space $(\Omega, F, P)$. Throughout this paper, we consider the following LSEPCAs:

$$
\begin{align*}
\frac{dx(t)}{dt} &= (Ax(t) + Bx([t]))dt + (Cx(t) + Dx([t]))dB(t), \quad t \in [0, T] \\
x(0) &= x_0
\end{align*}
$$

(1)

where $T > 0, A \in \mathbb{R}^{d\times n}$ the matrix [8], $x_0 \in C^{[0,T]}_{\mathbb{F}_t}(\mathbb{R}^n)$, $B, C, D$ are constants. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$
x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bx([s])ds + \int_0^t e^{A(t-s)}(Cx(s) + Dx([s]))dB(s) \quad \forall t \geq 0.
$$

(2)

Let $h = \frac{1}{m}$ be a given step size with integer $m \geq 1$; and let the grid points $t_n$ be defined by $t_n = nh(n = 0, 1, 2, \cdots)$. We consider the exponential Euler method to (1)
\[ y_{n+1} = e^{Ah}y_n + e^{Ah}\left(By^h([nh])\right)h + e^{Ah}\left(Cy^h + Dy^h([nh])\right)\Delta B_n, n = 0, 1, 2, \cdots \]  
(3)

where \( \Delta B_n = B(t_n) - B(t_{n-1}), n = 0, 1, 2, \cdots \), \( y_n \) is approximation to the exact solution \( x(t_n) \), \( y^h([nh]) \) is approximation to the exact solution \( x([nh]) \). Let \( n = km + l \) \( (k = 0, 1, 2, \cdots, l = 0, 1, 2, \cdots, m - 1) \), we obtain

\[ y_{km+l+1} = e^{Ah}y_{km+l} + e^{Ah}\left(By_{km}^h\right)h + e^{Ah}\left(Cy_{km+l}^h + Dy_{km+l}^h\right)\Delta B_{km+l} \]  
(4)

where \( \Delta B_{km+l} = B(t_{km+l}) - B(t_{km+l-1}) \), \( y_{km} \) is approximation to the exact solution \( x(t_{km}) \), \( y_{km+l} \) is approximation to the exact solution \( x(t_{km+l}) \). The continuous exponential Euler method approximate solution is defined by

\[ y(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} \left(Bz([s]) + Cz(s) + Dz([s])\right) dB(s) \]  
(5)

where \( z(t) = \sum_{k=0}^{m} y_{kh}I_{[kh,(k+1)h)}(t) \) with \( I_A \) denoting the indicator function for the set \( A \). It is not difficult to see that \( y(t_{km+l}) = z(t_{km+l}) = y_{km+l} \) for \( k = 0, 1, 2, \cdots, l = 0, 1, 2, \cdots, m - 1 \). That is, the step function \( z(t) \) and the continuous exponential Euler solution \( y(t) \) coincide with the discrete solution at the grid point.

**Exponential Stability in Mean Square**

In this section, we give the exponential stability in mean square of the exact solution and the exponential Euler method to linear stochastic differential equations with piecewise continuous arguments (1).

**Stability of the Exact Solution**

In this subsection, we will show the exponential stability in mean square of the exact solution to linear stochastic delay differential equations with piecewise continuous arguments (1).

**Theorem 3.1.** If \( 1 + 2\mu[A] + 4\left(B^2 + C^2 + D^2\right) < 0 \), then the solution of equations (1) with the initial data \( \xi \in C_{F_0}^b([-\tau, 0], R^n) \) is exponentially stable in mean square, that is,

\[ E|y(t)|^2 \leq m(1)^{-1}E|y(0)|^2 e^{\lim_{n \to 1}E|y(t)|^2}, t \geq 0 \]

where \( m(1) = e^{\beta_2} - \frac{\beta_2}{\beta_1} (1 - e^{\beta_1}) \), \( \beta_1 = 1 + 2\mu[a] + 2(B^2 + C^2 + D^2), \beta_2 = 2(B^2 + C^2 + D^2) \).

**Definition 3.1.[7]** LSEPCAs (1) are said to be exponentially stable in mean square if there is a pair of positive constants \( \lambda \) and \( \mu \) such that for any initial data \( x_0 \in C_{F_0}^b(R^n) \)

\[ E|y(t)|^2 \leq \mu E|x_0|^2 e^{-\lambda t}, t \geq 0. \]

We refer to \( \lambda \) as the rate constant and to \( \mu \) as the growth constant.

**Definition 3.2.[9]** The logarithmic norm \( \mu[A] \) of \( A \) is defined by

\[ \mu[A] = \lim_{\Delta \to 0} \frac{\|I + A\Delta\|^{-1}}{\Delta}. \]

Especially, if \( \| \cdot \| \) is an inner product norm, \( \mu[A] \) can also be written as

\[ \mu[A] = \max_{\xi \neq 0} \frac{<A\xi, \xi>}{\|\xi\|^2}. \]  
(6)
Lemma 3.1. Let \( m(t) = e^{\beta t} - \frac{\beta_2}{\beta_1} (1 - e^{\beta t}) \). If \( \beta_1 < 0, \beta_2 > 0 \) and \( \beta_1 + \beta_2 < 0 \) then for all \( t \geq 0, 0 < m(t) \leq 1 \).

Proof. It is known from \( \beta_1 < 0, \beta_2 > 0 \) and \( \beta_1 + \beta_2 < 0 \) then for all \( t \geq 0 \),
\[
m(t) = \frac{\beta_1 + \beta_2}{\beta_1} e^{\beta t} - \frac{\beta_2}{\beta_1} > 0
\]
and
\[
m(t) = e^{\beta t} - 1 + \frac{\beta_2}{\beta_1} (e^{\beta t} - 1) + 1 = \frac{\beta_1 + \beta_2 (e^{\beta t} - 1)}{\beta_1} + 1 \leq 1.
\]
The proof is complete.

Proof of Theorem 3.1. By Itô’s formula and Definition 3.2, for all \( t \geq 0 \); we have
\[
d \left| x(t) \right|^2 = \left( \langle 2x(t), Ax(t) + (Bx([t])) \rangle + \langle (Cx(t) + Dx([t])) \rangle \right) dt + 2x^T(t)(Cx(t) + Dx([t]))dB(t)
\]
\[
+ (\beta_1 \left| x([t]) \right|^2 + \beta_2 \left| x([t]) \right|^2) dt + 2x^T(t)(Cx(t) + Dx([t]))dB(t)
\]
where \( \beta_1 = 1 + 2\mu[A] + 2(B^2 + C^2 + D^2), \beta_2 = 2(B^2 + C^2 + D^2) \). Let \( V(x,t) = e^{-\beta t} \left| x(t) \right|^2, \) by Itô’s formula, we obtain
\[
d(e^{-\beta t} \left| x(t) \right|^2) = -\beta_1 e^{-\beta t} \left| x(t) \right|^2 dt + e^{-\beta t} d \left| x(t) \right|^2
\]
\[
\leq e^{-\beta t} \beta_2 \left| x([t]) \right|^2 dt + 2e^{-\beta t} x^T(t)(Cx(t) + Dx([t]))dB(t)
\]
Integrating (8) from \( t \) to \( [t] \) and taking expected values gives
\[
e^{-\beta t} E \left| x(t) \right|^2 \leq (e^{-\beta t(m)} - \frac{\beta_2}{\beta_1} (e^{-\beta t(m) - e^{-\beta t(m)}})) E \left| x([t]) \right|^2.
\]
So
\[
E \left| x(t) \right|^2 \leq (e^{-\beta t(m) - e^{-\beta t(m)}}) E \left| x([t]) \right|^2.
\]
For any \( t \in [k-1, k] \), we have
\[
E \left| x(t) \right|^2 \leq (e^{-\beta t(k-1)} - \frac{\beta_2}{\beta_1} (e^{-\beta t(k-1) - e^{-\beta t(k-1)}})) E \left| x([t]) \right|^2.
\]
Hence, we have
\[
E \left| x(k) \right|^2 = \lim_{t \to k} E \left| x(t) \right|^2 \leq m(1) E \left| x(k-1) \right|^2 \leq m(1)^k E \left| x(0) \right|^2.
\]
By (9) and Lemma 3.1, we obtain
\[
E \left| x(t) \right|^2 \leq m(1) E \left| x(k-1) \right|^2 \leq m(1)^k E \left| x(0) \right|^2
\]
\[
= e^{(k-1)ln m(1)} E \left| x(0) \right|^2 = e^{(k-1)ln m(1)} E \left| x(0) \right|^2 e^{ln m(1)}
\]
\[
\leq e^{-ln m(1)} E \left| x(0) \right|^2 e^{ln m(1)} = m(1)^{-1} E \left| x(0) \right|^2 e^{ln m(1)},
\]
which proves the theorem.

Stability of the Exponential Euler Method

In this subsection, under the same conditions as those in Theorem 3.1, we will obtain the exponential stability in mean square of the exponential Euler method (4) to LSEPCAs (1).
Definition 3.3. Given a step size $h = \frac{1}{m}$ for some positive integer m, the discrete exponential Euler method is said to be exponentially stable in mean square on SDDEs (1) if there is a pair of positive constants $\lambda$ and $\mu$ such that for any initial data $x_0 \in C_{b,0}^b(\mathbb{R}^n)$,

$$E|y_n|^2 \leq \mu |x_0|^2 e^{-\lambda nh}, n \geq 0.$$ 

Lemma 3.2. Let $\mu[A]$ be the smallest possible one-sided Lipschitz constant of the matrix $A$ for a given inner product. Then $\mu[A]$ is the smallest element of the set

$$M = \{\theta : \exp(\theta t) \leq \exp(\delta t), t \geq 0\}$$

Lemma 3.3. Let $M(t) = a'_i + a_2 (a'_i - 1) a_i^{-1} \cdot t > 0$. If $a_i > 0, a_2 > 0$ and $a_i + a_2 < 1$, then $0 < M(t) \leq 1$ for all $t > 0$.

Proof. By the conditions $a_i > 0, a_2 > 0$, and $a_i + a_2 < 1$, I have for all $t > 0$,

$$M(t) = a'_i + a_2 (a'_i - 1) a_i^{-1} = \frac{(a_i + a_2 - 1)(a'_i - 1)}{a_i - 1} + 1 \leq 1$$

which proves the lemma.

Theorem 3.2. If $1 + 2\mu[A] + 4\left(B^2 + C^2 + D^2\right) < 0$ then for all $h > 0$ the numerical method to equations (1) is exponentially stable in mean square, that is

$$E\left|y_{km+l}\right|^2 \leq M^{-1}(m)E\left|y_0\right|^2 e^{(km+l)h}M(m), k \geq 0$$

where $M(t) = a'_i + a_2 (a'_i - 1) a_i^{-1}$, $a_i = e^{\mu[A]h}(1 + B^2 + C^2 + D^2) h^2 + 2(B^2 + C^2 + D^2) h + h)$, $a_2 = e^{2\mu[A]h}(B^2 + C^2 + D^2) h^2 + 2(B^2 + C^2 + D^2) h)$.

Proof. Squaring and taking the conditional expectation on both sides of (7), noting that $\Delta B_{km+l}$ is independent of $F_{(km+l)h}$, we have

$$E\left(\Delta B_{km+l}F_{(km+l)h}\right) = E\left(\Delta B_{km+l}\right) = 0$$

and

$$E\left(\Delta B_{km+l}^2 \right) = h,$$ 

we have

$$E\left(\Delta B_{km+l}F_{(km+l)h}\right) \leq e^{2\mu[A]h}(1 + h)E\left|y_{km+l}\right|^2 + e^{2\mu[A]h}hE\left|y_{km+l}\right|^2 \left|F_{(km+l)h}\right|$$

Taking expectations on both sides, we obtain that

$$E\left|y_{km+l}\right|^2 \leq a_1 E\left|y_{km+l}\right|^2 + a_2 E\left|y_{km}\right|^2$$

Where

$$a_1 = e^{2\mu[A]h}(1 + B^2 + C^2 + D^2) h^2 + 2(B^2 + C^2 + D^2) h + h),$$

$$a_2 = e^{2\mu[A]h}(B^2 + C^2 + D^2) h^2 + 2(B^2 + C^2 + D^2) h).$$

for all $h > 0$; which implies

$$1 + h + 4(B^2 + C^2 + D^2) h + 2(B^2 + C^2 + D^2) h^2 < 1 - 2\mu[A]h + \frac{(2\mu[A]h)^2}{2!} < e^{-2\mu[A]h}.$$

That is $a_1 + a_2 < 1$ for all $h > 0$. for all $n = 1, 2, \ldots$, we have

$$E\left|y_{km+l}\right|^2 \leq (a_1^{l+1} + a_2^{l+1} a_1^{-1}) E\left|y_{km}\right|^2 = M(l+1) E\left|y_{km}\right|^2$$

when $l = m-1$,

$$E\left|y_{km+l}\right|^2 \leq M(m)E\left|y_{km}\right|^2 = M(m)^{k+1} E\left|y_0\right|^2$$

By Lemma 3.3., we have
\[ E|y_{km+1}|^2 \leq M(l+1)M(m)^k E|y_0|^2 \leq M(m)^k E|y_0|^2 \]
\[ \leq e^{k \ln M(m)} E|y_0|^2 = e^{k \ln M(m)} E|y_0|^2 \]
\[ = e^{-(l+1)\ln M(m)} e^{(km+l+1)\ln(m)} = e^{-(l+1)\ln M(m)} E|y_0|^2 e^{(km+l+1)\ln(m)} \]
\[ = M^{-1}(m) E|y_0|^2 e^{(km+l+1)\ln(m)} \]

The proof is completed.

**Numerical Experiments**

In this section, we give numerical experiment in order to demonstrate the results about the exponential stability in mean square of the numerical solution for equations (1). We consider the test equation

\[ dx(t) = (a_1 x(t) + a_2 x([t])) dt + (b_1 x(t) + b_2 x([t])) dB(t) \quad \forall t \geq 0 \tag{10} \]

**Example 4.1.** When \( a_1 = -2, a_2 = 1, b_1 = 0.3, b_2 = 0.1, x_0 = 1 \) We can show the stability of the exponential Euler method to (3). In Figure 1, all the curves decay toward to zero when \( h = \frac{1}{2}, h = \frac{1}{4}, h = \frac{1}{8}, h = 1 \). So we can consider that our experiments are consistent with our proved results in Section 3.

![Figure 1](image_url)

Figure 1. When \( h = \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1 \), Numerical Solutions exponential stability in mean square.

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**References**


