On the Generalized Shift-HSS Splitting Methods for Nonsingular Saddle Point Problem

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Abstract. In this paper, we establish a generalized shift-HSS splitting (denoted by GSFHSS) iteration method for solving nonsymmetric saddle point system where the (1,1)-block sub-matrix is nonsymmetric positive definite and investigate that the convergence property of the GSFHSS iteration method.

Introduction

We consider the nonsymmetric saddle point linear system of form:

\[ Kx = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \]

(1)

where \( A \in \mathbb{R}^{n \times n} \) is nonsymmetric positive definite, that is, its symmetric part \( \frac{1}{2}(A + A^T) \) is positive definite, \( B \in \mathbb{R}^{n \times m} \) (\( n \geq m \)) has full column rank (i.e. \( \text{rank}(B) = m \)) with \( B^T \) denoting the transpose of \( B \), and \( f \in \mathbb{R}^n \) and \( g \in \mathbb{R}^m \) are given vectors.

This class of linear systems of the form (1) arises in a variety of scientific and engineering applications, including interior point methods in constrained optimization, constrained least squares problems and generalized least squares problems, optimization, incompressible flow problems, and so on. For an overview of the many applications, we refer the reader to [1] for more discussion on this subject.

Lately, Bai and Yin [6] established the shift-splitting iteration method for solving the saddle point system \( Kx = b \), where the large sparse non-Hermitian positive definite coefficient matrix \( K \in \mathbb{R}^{n \times n} \), in other words, its Hermitian part \( \frac{1}{2}(K + K^H) \) is Hermitian positive definite with \( K^H \) denoting the conjugate transpose of \( K \). Based on the idea of the shift-splitting [6], Cao et al. [7] applied the shift-splitting iteration method to solve the saddle point system (1) and further presented a local shift-splitting preconditioner.

On the GSFHSS Iteration Method

In this section, we consider another splitting for the coefficient matrix of the saddle point problem (1):

\[ K = 2M_{\text{GSFHSS}} - 2N \]

\[ = \begin{pmatrix} \frac{1}{\alpha}(\alpha I_n + H)(\alpha I_n + S) & B \\ -B^T & \beta I_m \end{pmatrix} - \begin{pmatrix} \frac{1}{\alpha}(\alpha I_n - H)(\alpha I_n - S) & -B \\ B^T & \beta I_m \end{pmatrix} \]

(2)

with the positive iteration parameters \( \alpha \) and \( \beta \).

For convenience, we denote the GSFHSS iteration matrix \( \tau(\alpha, \beta) \) by
Let $\lambda$ be an eigenvalue of the iteration matrix $\tau(\alpha, \beta)$ in (3) and $(u^*, v^*)^*$ be the corresponding eigenvector with $u \in C^n$ and $v \in C^n$. We consider the following generalized eigenvalue problem:

$$ N \begin{pmatrix} u \\ v \end{pmatrix} = \lambda M_{\text{GSFHSS}} \begin{pmatrix} u \\ v \end{pmatrix}. $$

(4)

After some algebra, the generalized eigenvalue problem (4) is equivalent to the following form:

$$ \begin{align*}
\frac{1}{\alpha} (\alpha I_n - H)(\alpha I_n - S)u - Bv &= \frac{1}{\alpha} \lambda (\alpha I_n + H)(\alpha I_n + S)u + \lambda Bv, \\
B^T u + \beta v &= -\lambda B^T u + \beta \lambda v.
\end{align*} $$

(5)

We give the following lemmas to investigate the convergence property of the GSFHSS iteration method:

**Lemma 2.1** ([8]). If $S$ is a skew-Hermitian matrix, then $iS$ ($i$ is the imaginary unit) is a Hermitian matrix and $u^* Su$ is a purely imaginary number or zero for all $u \in C^n$.

**Lemma 2.2** ([9]). Both roots of the complex quadratic equation $\lambda^2 - \Phi \lambda + \Psi = 0$ have modulus less than one if and only if $|\Phi - \Phi| + |\Psi| < 1$, where $\Phi$ denotes the conjugate complex of $\Phi$.

**Lemma 2.3.** Let $\lambda$ be an eigenvalue of $\tau(\alpha, \beta)$ with $\alpha, \beta > 0$. Then $\lambda \neq 0$.

**Proof.** Assume $\lambda = 0$, then the equation (4) is reduced to the following form

$$ \begin{pmatrix} A & B \\
-B^T & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. $$

Since $K$ is nonsingular matrix, then it is easy to see that $u = 0$ and $v = 0$. This contradicts with the assumption that $(u^*, v^*)^*$ is an eigenvector, therefore, $\lambda \neq 0$.

So, this proof is completed. □

**Lemma 2.4.** Let $A$ be a nonsymmetric positive definite and $B$ have full column rank. Assume that $\lambda$ is an eigenvalue of the iteration matrix $\tau(\alpha, \beta)$ as defined in (3) with $\alpha, \beta > 0$, and $(u^*, v^*)^*$ is the corresponding eigenvector with $u \in C^n$ and $v \in C^n$, if $0 \neq u \in \mathbb{R}(B^T)$, then $|\lambda| < 1$.

**Proof.** It is easy to verify that $u \neq 0$. Unless, if $u = 0$, by the second of the equations (5), we know that $B(\lambda - 1)v = 0$. Following Lemma 2.3 and by the use of $\lambda \neq 1$, then $Bv = 0$. As $B$ has full column rank, then, we further conclude $v = 0$, which contradicts with the assumption that $(u^*, v^*)^*$ is an eigenvector. So $u \neq 0$.

If $0 \neq u \in \mathbb{R}(B^T)$, for convenience, we assume that $\|u\|_2 = 1$. Following the second of the equations (5), we get that $v = 0$. From the proof of [10, Theorem 2.2], multiply the first of the equations (5) from the left-hand side by $u^*$, we can easy to see that

$$ |\lambda| \leq \| (\alpha I_n - H)(\alpha I_n + H)^{-1} \|_2 < 1. $$

This completes the proof.

**Theorem 2.1.** Let the conditions of Lemma 2.4 be satisfied. Assume that $\lambda$ is an eigenvalue of the iteration matrix $\tau(\alpha, \beta)$ as defined in (3) and denote

$$ \begin{align*}
\frac{u^* Au}{u u} &= a + bi, \\
\frac{u^* HSu}{u u} &= c + di \quad \text{and} \quad \frac{u^* BB^T u}{u u} = e,
\end{align*} $$

where $a, e > 0$,

$$ c = \frac{1}{2} \frac{u^* (HS - SH) u}{u u} \quad \text{and} \quad d = \frac{1}{2i} \frac{u^* (HS + SH) u}{u u}. $$

If the positive iteration parameter $\alpha$ and $\beta$ satisfy the following inequality

$$ |\lambda| \leq \| (\alpha I_n - H)(\alpha I_n + H)^{-1} \|_2 < 1. $$

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Then the GSFHSS iteration method is convergence, that is, 
\[ |\lambda| < 1. \]

**Proof.** From Lemma 2.3 and Lemma 2.4, we only need to consider the case \( B^T u \neq 0 \). If \( u \notin \mathbb{R}(B^T) \), according to the second of the equations (5), we have that

\[ v = \frac{\lambda + 1}{\beta(\lambda - 1)} B^T u. \]  

(8)

Substitute it into the first of the equations (5), by straightforward computation, it is easy to see that

\[(\alpha I_n - H)(\alpha I_n - S)u = \lambda(\alpha I_n + H)(\alpha I_n + S)u + \frac{\alpha(\lambda + 1)^2}{\beta(\lambda - 1)} BB^T u. \]

(9)

Multiplying the equation (9) from the left-hand side by \( u^* \), after straightforward calculations, then the equation (8) yields the following form

\[ \alpha^2(\lambda - 1)^2 + \alpha(\lambda^2 - 1)\frac{u^* Au}{u^* u} + (\lambda - 1)^2\frac{u^* HSu}{u^* u} + \frac{\alpha(\lambda + 1)^2}{\beta} \frac{u^* BB^T u}{u^* u} = 0. \]

(10)

By the use of (6) and denote \( \tilde{e} = \frac{\alpha e}{\beta} \), we know that \( \lambda \) satisfies the following real quadratic equation:

\[ [(\alpha^2 + \alpha a + c + \tilde{e} + (\alpha b + d)i)\lambda^2 + 2(\tilde{e} - \alpha^2 - c - di)\lambda + \alpha^2 - \alpha a + c + \tilde{e} - (\alpha b - d)i = 0. \]

If \( \alpha^2 + \alpha a + c + \tilde{e} + (\alpha b + d)i = 0 \), then know that \( \alpha^2 + \alpha a + c + \tilde{e} = 0 \) and \( \alpha b + d = 0 \). So we have

\[ \lambda = -\frac{\alpha^2 - \alpha a + c + \tilde{e} - (\alpha b - d)i}{2(\tilde{e} - \alpha^2 - c - di)} = \frac{\alpha a + abi}{2\tilde{e} + \alpha a + abi}. \]

Since \( \alpha, a, \tilde{e} > 0 \), then we can easily obtain that

\[ |\lambda| = \frac{(\alpha a)^2 + (\alpha b)^2}{(2\tilde{e} + \alpha a)^2 + (\alpha b)^2} < 1. \]

In order to verify the convergence property, we consider the case \( \alpha^2 + \alpha a + c + \tilde{e} + (\alpha b + d)i \neq 0 \). By Lemma 2.2, it is easy to see that \( |\lambda| < 1 \) if and only if \( |(\Phi - \bar{\Phi})\Psi| + |\Psi|^2 < 1 \). Without loss of generality, denote \( \Phi \) and \( \Psi \) by

\[ \Phi = \frac{2(\tilde{e} - \alpha^2 - c - di)}{\alpha^2 + \alpha a + c + \tilde{e} + \alpha b + d)} \quad \text{and} \quad \Psi = \frac{\alpha^2 - \alpha a + c + \tilde{e} - (\alpha b - d)i}{\alpha^2 + \alpha a + c + \tilde{e} + (\alpha b + d)i}. \]

A simple computation reveals that

\[ |(\Phi - \bar{\Phi})\Psi| + |\Psi|^2 = \frac{4(\Theta + (2\tilde{e}d)^2 + (\alpha^2 - \alpha a + c + \tilde{e})^2 + (\alpha b - d)^2}{(\alpha^2 + \alpha a + c + \tilde{e})^2 + (\alpha b + d)^2} \]

where \( \Theta = (\alpha^2 - \alpha^2 a - ac - bd)^2 \).

If the following inequality is holds

\[ 4\alpha \tilde{e} \alpha^2 (\alpha^2 a + ac + bd) > (2\tilde{e} d)^2, \]

by straightforward computation, we have the inequality of form

\[ 4\alpha \tilde{e} \alpha^2 (\alpha^2 a + ac + bd) > (2\tilde{e} d)^2, \]

(11)
\[
|\Phi - \bar{\Phi}\Psi| + |\Psi| \leq \\
\frac{4\sqrt{\Theta + 4a\bar{a}^2(\alpha^2 a + ac + bd) + (\alpha^2 - \alpha a + c + \bar{e})^2 + (\alpha b - d)^2}}{\gamma(\alpha^2 + \alpha a + c + \bar{e})^2 + (\alpha b + d)^2} \\
= \frac{4\alpha(\alpha \bar{e} + \alpha^2 a + ac + bd) + (\alpha^2 - \alpha a + c + \bar{e})^2 + (\alpha b - d)^2}{\gamma(\alpha^2 + \alpha a + c + \bar{e})^2 + (\alpha b + d)^2} \\
= 1.
\]

Following the inequality (11), after some algebra, we complete this proof.

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References