A New Reynolds Number (Re) Criterion for Using Newtonian/non-Newtonian Constitutive Models in Blood Flow Simulations

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Abstract. A circular cylindrical tube is employed as the standard vessel model and blood flows through it are simulated respectively by using the Newtonian and four non-Newtonian constitutive models. Results including velocity profiles, flow rates and wall shear stress (WSS) are emphatically analyzed to show differences between the Newtonian and non-Newtonian models. When Re is low (Re<100), all non-Newtonian velocity profiles are more flat at the tube central region and with higher gradients near the wall than the Newtonian profile. As Re increases, the non-Newtonian profile gradually shrinks to the Newtonian. The non-Newtonian flow rate first varies nonlinearly with the low pressure gradient (dp/dx<200Pa/m) and then trends linearly with the high pressure gradient. When Re>1000, relative WSS variations of the Newtonian model to non-Newtonian models can decrease down below 20%, which concludes that Re>1000 can be regarded as a new criterion to determine whether the Newtonian assumption is suitable for the blood flow simulation. The Re criterion is more feasible in practice than the previous shear rate criterion (>100s⁻¹), because the shear rate is always hard to know specifically before implementing the simulation.

Keywords: non-Newtonian model, Reynolds number, blood flow, CFD

1 Introduction

Recent years, computational fluid dynamics (CFD) has become a very useful technique to investigate hemodynamics in the human cardiovascular system, which can give amounts of information including blood flow patterns, wall pressure gradient (WPG) and WSS in a low-cost and non-invasive way [1]. In CFD studies, the blood is always assumed to be Newtonian with a constant viscosity. Tse et al. [2] solved the blood flow in a patient-specific dissecting aneurismal aorta by using the Newtonian constitutive model. Wu et al. [3] simulated transient blood flows through partially occluded curved arteries with a fixed viscosity. However, the real blood is a multi-component mixture consisting of the plasma, blood cells, platelets, etc. and possesses the shear-thinning behavior that the viscosity decreases as the shear rate increases [4]. Currently, many constitutive models fitting experimental data have been developed to describe the non-Newtonian property of blood, such as the Casson model, Carreau-Yasuda model and Powell-Eyring model [5]. Apostolidis et al. [6] computed blood flows in the coronary artery by the Casson model and showed significant differences (up to 50%) between Newtonian and non-Newtonian results. Soares et al. [7] also pointed out that the Newtonian model predicted the smaller WSS amplitude at the abdominal aorta bifurcation than the non-Newtonian model did.

Although the non-Newtonian constitutive model can reflect rheological behaviours of the real blood to give more accurate CFD results, it also brings computational complexities in the preliminary research. In fact, the Newtonian assumption is valid enough where the shear rate of blood flow is greater than 100s⁻¹ [8]. However, the shear rate criterion is not so practical, because one cannot know the specific flow shear rate before implementing the CFD simulation. Additionally, cardiovascular organs studied in previous literatures are always different. Therefore, it is very difficult to give clear and general comparison conclusions between the non-Newtonian and Newtonian models.

To solve aforementioned problems, this work first introduces a circular cylindrical tube as the standard model of the blood vessel and gives governing equations of the blood flow. Then, blood flows through the tube are simulated respectively by using Newtonian and four non-Newtonian models. Finally, by comparisons between Newtonian and non-Newtonian results, this work clarifies non-Newtonian effects in the tube blood flow and obtains a new practical Reynolds number criterion to determine under what conditions the Newtonian model is suitable for use.

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2 The Blood Vessel Model and Governing Equations of the Blood Flow

A circular cylindrical tube is employed as the standard model of the blood vessel. As shown in Fig.1, x-axis is along the tube centreline and $D$ is the tube diameter. The tube blood flow is assumed to be fully developed and an infinitesimal cylinder of the tube with the radius $r$ and length $dx$ is investigated. $\tau$ and $p$ are the viscous shear stress and pressure, respectively. Implementing the force balance, there is

$$\tau \cdot 2\pi r dx = [p(x + dx) - p(x)] \pi r^2$$

(1)

The shear stress follows

$$\tau = \mu(\dot{\gamma}) \cdot \frac{du}{dr}$$

(2)

where $\mu$ is the blood dynamic viscosity, and $\dot{\gamma}$ is the shear rate. If the viscosity $\mu$ is constant, the blood is Newtonian. From Eqs.(1) and (2), it can deduce

$$\mu \cdot \frac{du}{dr} = \frac{r \cdot dp}{2 \cdot dx}$$

(3)

Eq.(3) is the basic governing equation of the tube blood flow. On the wall, there is the no-slip boundary condition as follows:

$$u |_{r=2} = 0$$

(4)

Figure 1. The schematic of the circular cylindrical tube.

Four non-Newtonian constitutive models are used to solve Eq.(3) respectively for comparison, that are the Casson model, Carreau-Yasuda model, modified Powell-Eyring model and modified Cross model [5]. Specific expressions and parameters of each model are list in Tab.1. Fig.2 compares blood viscosities given by all four models. All non-Newtonian models can reflect the shear-thinning effect of the real blood. As the shear rate increases, viscosities of non-Newtonian models all shrink to the Newtonian value $\mu_n=0.00345$ Pa·s. When the shear rate becomes greater than 100s$^{-1}$, differences between Newtonian and non-Newtonian viscosities are very small.

In this work, the governing equation (3) combined with the boundary condition (4) is solved numerically by the fourth-order Runge-Kutta method [9], in which the velocity $u$ is treated as the primitive variable and the pressure gradient $dp/dx$ is specifically given. In the solution process, the velocity will be calculated by marching from the tube wall to the centerline. Once the velocity profile is obtained, the volumetric flow rate $Q$ and Re can be calculated subsequently. The current Re is defined as:

$$Re = \frac{\rho U_m D}{\mu}$$

(5)

where $U_m$ is the mean flow velocity through the tube and $\mu_a$ is the blood apparent viscosity calculated by

$$\mu_a = \frac{\pi D^4}{128Q} \cdot \frac{dp}{dx}$$

(6)

<table>
<thead>
<tr>
<th>Model</th>
<th>Expression</th>
<th>Parameters</th>
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<tbody>
<tr>
<td>Casson</td>
<td>$\sqrt{\mu} = \sqrt{\mu_n + \sqrt{\gamma}}$</td>
<td>$\mu_n = 0.00345$ Pa·s, $\tau_n = 0.005$ Pa</td>
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<tr>
<td>Carreau-Yasuda</td>
<td>$\mu = \mu_n + (\mu_n - \mu_a) \times \left[1 + (\dot{\gamma})^\alpha \right]^{\beta-1}$</td>
<td>$\mu_n = 0.00345$ Pa·s, $\mu_a = 0.056$ Pa, $\lambda = 1.902$s, $a = 1.25, \alpha = 0.22$</td>
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<tr>
<td>Modified Powell-Eyring</td>
<td>$\mu = \mu_n + (\mu_n - \mu_a) \times \frac{\ln(1 + \lambda \dot{\gamma})}{(\lambda \dot{\gamma})^m}$</td>
<td>$\mu_n = 0.00345$ Pa·s, $\mu_a = 0.056$ Pa, $\lambda = 2.415$s, $m = 10.89$</td>
</tr>
<tr>
<td>Modified Cross</td>
<td>$\mu = \mu_n + (\mu_n - \mu_a) \times \frac{1}{[1 + (\dot{\gamma})^\alpha]^{m}}$</td>
<td>$\mu_n = 0.00345$ Pa·s, $\mu_a = 0.056$ Pa, $\alpha = 3.736$, $m = 2.406, \alpha = 0.254$</td>
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Table 1. Non-Newtonian models used in this work.

Figure 2. Blood viscosities calculated by different non-Newtonian models.

3 Results

In this section, the tube blood flow is assumed to be laminar. Particularly, when the blood is Newtonian, the solution of Eq.(3) is well-known as the Poiseuille flow with a parabolic velocity profile. Numerical results including velocity profiles, flow rates and WSS are emphatically analyzed.

3.1 Velocity

Fig.3 shows blood flow velocity profiles of different Re numbers respectively given by four non-Newtonian constitutive models, in which $R$ is the tube radius and $U_0$ is the maximum flow velocity. The Poiseuille profile (the Newtonian velocity profile) is also plotted for
comparison. When Re is low, such as Re=10 and Re=100, all non-Newtonian profiles are more flat at the tube central region and with higher velocity gradients near the tube wall than the Poiseuille profile. As Re grows, non-Newtonian profiles gradually shrink to the Poiseuille profile. When Re>1000, all non-Newtonian profiles become very close to the Poiseuille parabola.

Fig.4 compares velocity profiles of different models with same Reynolds numbers, in which $U_m$ is the mean blood flow velocity. Four non-Newtonian constitutive models give similar profiles at the same Reynolds number. Regardless of the Reynolds number, the ratio of the maximum velocity to the mean velocity of the Poiseuille flow remains 2. However, at Re=10, all non-Newtonian maximum velocity ratios are much lower than the Newtonian value, only around 1.7. As Re increases, all non-Newtonian profiles gradually tend toward the Poiseuille profile and maximum velocity ratios reach close to 2.

Figure 3. Blood flow velocity profiles of different Re numbers: (a) Casson model; (b) Carreau-Yasuda model; (b) modified Powell-Eyring model; (d) modified Cross model.

Figure 4. Blood flow velocity profiles of different models: (a) Re=10; (b) Re=100; (c) Re=500; (d) Re=1000.
3.2 Flow Rate

Fig. 5 plots variations of blood flow rates with the pressure gradient predicted by the Newtonian and non-Newtonian models. For the Newtonian case, the famous Poiseuille law describes a linear relation between the flow rate and pressure gradient as follows:

\[ Q = \frac{\pi D^4}{128\mu} \frac{dp}{dx} \]

which can clearly reflect the WSS difference between the Newtonian and non-Newtonian models. Apparently, the greater the Re number is, the smaller the WSS difference becomes. Particularly, when Re>1000, relative WSS variations of the Carreau-Yasuda and modified Powell-Eyring models decrease down below 20%, which can deduce that the simulation by using the Newtonian model is reasonable for the tube blood flow with Re>1000. Therefore, Re>1000 can be regarded as a more feasible criterion to determine whether the Newtonian assumption is suitable for the blood flow simulation through the artery or vein, to replace the criterion of \( \gamma > 100 \text{s}^{-1} \). The shear rate is always hard to know specifically before carrying our CFD simulations.

3.3 WSS

Previous literatures paid great attention to WSS data which may contribute to developments of some diseases in clinical. Fig. 6 illustrates relative variations of the Newtonian WSS to the non-Newtonian. The relative WSS variation is defined as follows:

\[ \frac{\Delta WSS}{WSS} = \left| \frac{WSS_{\text{Newton}} - WSS_{\text{non-Newtonian}}}{WSS_{\text{Newton}}} \right| \]

Conclusions

This work simulates the blood flow through a circular cylindrical tube, which is treated as the standard model of the blood vessel. In simulations, the Newtonian and four non-Newtonian constitutive models are used respectively. Blood flow results including velocity profiles, flow rates and WSS are elaborately compared to show differences between the Newtonian and non-Newtonian models.

When Re<100, non-Newtonian velocity profiles are more flat at the tube central region and with higher gradients near the wall than the Newtonian Poiseuille profile. While Re>1000, all non-Newtonian profiles are very close to the Poiseuille profile. As the pressure gradient increases, the non-Newtonian flow rate varies from nonlinearly to linearly. At Re>1000, relative WSS variations of the Newtonian model to two non-Newtonian models (the Carreau-Yasuda and modified Powell-Eyring
models) decrease down below 20%, which concludes that Re>1000 can be used as a new criterion for employing the Newtonian assumption to simulate the blood flow through the artery or vein. Undoubtedly, the Re criterion recommended in this work is more practical than the previous shear rate criterion (\(\dot{\gamma} > 100 \text{s}^{-1}\)).

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**References**


