A Method of the High-close-order Gear Construction and Its Characteristics

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Keywords: High-order-close gear, Tooth profile, Smooth curvature connection, Concave-convex meshing

Abstract: This paper introduces the piecewise involute method to construct the High-order-close gear tooth profile. Also, parameter equation of the gear profile is obtained. Based on this method, it is proved that the profile curve is fourth-order derivative and continuous at connection points. It indicates that the curvature radius curvature and the Hertz contact stress mutation among the tooth profile won’t occur at the connecting points. The ranges of the initial parameters used to construct the gear are obtained based on the principles of the involute gear parameter standard. At last, a sequence of adequate parameters is selected to complete the solid model construction of the gear which finally achieve the concave-convex meshing form and the gear meshing behavior is analyzed.

Introduction

As is known, involute gear is one of the most important transmission forms in the modern mechanical transmission structures. However, with the increasing speed and high power of the mechanical operation, the small curvature radius at the meshing point because of the convex-convex contact form of the involute gear causes a relatively large contact stress, so that the involute gear carrying capacity is limited, and it often cause surface pitting and crack and other failure forms in the actual production. In addition, the involute gears sliding exists in the meshing process except for the nodes, and there is larger sliding movement in the tooth top and the tooth root. So, it is difficult to realize the pure rolling contact. American Wildhaber made a circular gear tooth profile in 1926 [1]. In 1958, the former Soviet scholar Novikov proposed the end of the circular arc tooth point meshing transmission theory to improve the carrying efficiency [2]. In the late 1980s, the Japanese scholar Xiao Shoumian proposed logic gear [3], which improved the carrying capacity of the gear, but he did not give the specific construction process of the gear. Zhao Han [4] and others put forward the micro-line tooth profile, and introduced its formation principle. Professor Liu Huran and others [5] applied the concept of "tangent" of differential geometry to the meshing of the gears to achieve the concavo-convex meshing contact of the tooth surface, which improved the contact state of the tooth surface and the strength of the gear. In this paper, based on the close contact theory and construction method of micro-segment involute gear tooth profile, the construction method of four-segment involute close gear is proposed, the tooth profile equation and the ranges of and the initial parameters are given for the new tooth. The research of the profile provides a new way of thinking.

Construction of High-close-order Gear Profile

Fundamental Principle

The high-order-close gear combines the advantages of involute tooth profile and circular tooth profile. The top part of the tooth is convex, and the root part is concave, that realizes the concave-convex meshing between a pair of gears. Contact stress and bending stress are reduced. According to the Hertz contact theory, when the two tooth profiles are in the form of
concave-convex meshing, the intermeshing tooth surface induced curvature is zero, that means the contact strength is maximal, and the curvature center of tooth profile must be on the pitch line.

Curve of curvature center is called evolute, and the corresponding curve is called involute. But for straight line, evolute and involute do not exist [6]. The multi-segment curves, whose curvature centers of the first and end of curve segments are on the same straight line, can be artificially constructed to satisfy the condition.

As shown in Fig. 1, the given point \( M_0(x_0, y_0) \), the corresponding curvature center coordinate \( x_{c0} \) and the direction of the tangent direction of the curve \( \beta \). Any curve is selected at point \( M_0(x_0, y_0) \) who is at gear profile, and it has the same coordinates, the corresponding curvature center coordinate \( x_{c0} \) and the tangential direction of the curve as the required tooth profile. Another point \( M_1(x_1, y_1) \) on the curve whose curvature center is not on the X-axis. Establish a new coordinate system \((O-X_1Y_1)\) by point \( M_1 \), and take following coordinate transformation.

\[
\begin{pmatrix}
X \\
Y
\end{pmatrix} = 
\begin{pmatrix}
\cos \beta & \sin \beta \\
-\sin \beta & \cos \beta
\end{pmatrix} \begin{pmatrix}
x - x_1 \\
y - y_1
\end{pmatrix}
\]  
\hspace{2cm} (1)

\[\begin{pmatrix}
X \\
Y
\end{pmatrix} = 
\begin{pmatrix}
-x \\
y
\end{pmatrix} = 
\begin{pmatrix}
-\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix} \begin{pmatrix}
x - x_1 \\
y - y_1
\end{pmatrix}
\]  
\hspace{2cm} (2)

\[\begin{pmatrix}
x \\
y
\end{pmatrix} = 
\begin{pmatrix}
x_1 \\
y_1
\end{pmatrix} + 
\begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix} \begin{pmatrix}
x - x_1 \\
y - y_1
\end{pmatrix}
\]  
\hspace{2cm} (3)

\[\begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix} = 
\begin{pmatrix}
x_1 \\
y_1
\end{pmatrix} + 
\begin{pmatrix}
-\cos 2\beta & -\sin 2\beta \\
-\sin 2\beta & \cos 2\beta
\end{pmatrix} \begin{pmatrix}
x - x_1 \\
y - y_1
\end{pmatrix}
\]  
\hspace{2cm} (4)

\[\begin{align*}
\bar{x} &= y_1 - \sin 2\beta (x_1 - x) + \cos 2\beta (y_1 - y) = 0 \\
\bar{y} &= x_1 - \cos 2\beta (x_1 - x) - \sin 2\beta (y_1 - y) = x_2
\end{align*}
\]  
\hspace{2cm} (5)

The corresponding \( x \) is calculated by formulas above and \( x_c \) and \( y_c \) are functions of \( x \), and get the
curve equation between \( M_1 \) and \( M_2 \).

\[
\begin{pmatrix}
  x_2 \\
  y_2
\end{pmatrix} = \begin{pmatrix}
  x_1 \\
  y_1
\end{pmatrix} + \begin{pmatrix}
  -\cos 2\beta & -\sin 2\beta \\
  -\sin 2\beta & \cos 2\beta
\end{pmatrix} \begin{pmatrix}
  x - x_1 \\
  y - y_1
\end{pmatrix}
\]

(6)

Set \( M_2 \) as the starting point, and repeat the steps above to construct the next segment of the curve until completing the curve of contacting media tooth profile.

**Construction Process**

Theoretically, any curve can be used for construction. Involute is used in construction because of its excellent nature. As shown in Fig.2, point \( O_1 \) is the base circle center of involute, and \( a \) is the radius length of the base circle, so we can get the parameter equation of involute segment \( AB \) in \((O_1-X_1Y_1)\).

\[
\begin{aligned}
  x &= a (\sin t - t \cos t) \\
  y &= a (\cos t + t \sin t)
\end{aligned}
\]

(7)

Take the derivative of the formula above, then we can get the slope of any point on the involute \( k = \frac{y'}{x'} = \cot(t) \). In addition, the curvature center coordinate of any point is \((X,Y) = (a \cos(t), a \sin(t))\), and this meets the requirements of the contact condition because all the curvature centers are on the base circle. Take two points \( A(x_A,y_A) \) and \( B(x_B,y_B) \) on involute, and make two normals separately at point \( A \) and point \( B \). Two intersections with base circle are \( A_1 \) and \( B_1 \), and corresponding generating angles are \( t_1 \) and \( t_2 \).

![Figure 2. Construction of the first two involutes.](image)

The second-segment involute \( BC \) is a standard involute in \((O_2-X_2Y_2)\), and the generating angle is \( u \), and radius of base circle is also \( a \). \( BC \) and \( AB \) are symmetrical about the straight line \( BB_1 \). \((O_2-X_2Y_2)\) can be seen as \((O_1-X_1Y_1)\) that has been rotated and displaced. \([M_{21}]\) is the rotation matrix and \( \vec{r}_{p2} = (x_{p2}, y_{p2}) \) is the translation vector. Then coordinate transformation equation between \((O_2-X_2Y_2)\) and \((O_1-X_1Y_1)\) can be obtained.

\[
\vec{r}_2 = [M_{21}](\vec{r}_i - \vec{r}_0)
\]

\[
\begin{pmatrix}
  x_2 \\
  y_2
\end{pmatrix} = \begin{pmatrix}
  x \\
  y
\end{pmatrix} = \begin{pmatrix}
  \cos \beta & -\sin \beta \\
  \sin \beta & \cos \beta
\end{pmatrix} \begin{pmatrix}
  \cos \beta & -\sin \beta \\
  \sin \beta & \cos \beta
\end{pmatrix}
\]

(8)

\( \beta \) is the angle of anticlockwise rotation, \( \beta = \pi - t_2 \). So the parameter equation of the second involute segment in \((O_1-X_1Y_1)\) can be obtained.

\[
\begin{aligned}
  x &= a [\sin(t-2u) + 2 \sin u - t \cos(t-2u)] \\
  y &= a [-\cos(t-2u) + 2 \cos u + t \sin(2u - t)]
\end{aligned}
\]

(9)

Set \( t = u - \Delta x (\Delta x \to 0^+) \) at the right end of the first involute segment and \( t = u + \Delta x (\Delta x \to 0^-) \) at the left.
end of the second involute segment. According to the results in Tab.1, it can be seen that the two symmetric involute segments are unconditionally fourth order continuous and derivable.

Table 1. Continuous derivative calculation results at point B.

<table>
<thead>
<tr>
<th>Left end point $t = u - \Delta x$</th>
<th>Right end point $t = u + \Delta x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dy}{dx}</td>
<td>_{x=x_0} = \frac{dy}{dx}</td>
</tr>
<tr>
<td>$\frac{d^2y}{dx^2}</td>
<td>_{x=x_0} = \frac{d^2y}{dx^2}</td>
</tr>
<tr>
<td>$\frac{d^3y}{dx^3}</td>
<td>_{x=x_0} = \frac{d^3y}{dx^3}</td>
</tr>
<tr>
<td>$\frac{d^4y}{dx^4}</td>
<td>_{x=x_0} = \frac{d^4y}{dx^4}</td>
</tr>
</tbody>
</table>

Based on the coordinate system $(O_2-X_2Y_2)$, a new coordinate system $(O_3-X_3Y_3)$ is established for the third involute segment $CD$. It should be noted that the third and fourth segment of the involute are on the other side of the pitch line, and the convexity is opposite to the first two involute segments. As shown in Fig.3, the third involute segment is also in coordinate system $(O_3-X_3Y_3)$. Its base circle radius is $r$, and rotation angle is $\phi$, and the standard equation in $(O_3-X_3Y_3)$ can be obtained.

\[
\begin{align*}
  x_3 &= -r \cos \phi - ru \sin \phi \\
  y_3 &= r \sin \phi - ru \cos \phi
\end{align*}
\] (10)

![Figure 3. Construction of the second and the third segment involutes.](image)

$[M_{1p}]$ and $[M_{p3}]$ are rotation matrices, and $\phi$ is rotation angle, and $\vec{t} = (x_s, y_s)$ is the translation vector. Then coordinate transformation equation between $(O_2-X_2Y_2)$ and $(O_3-X_3Y_3)$ can be obtained.

\[
\vec{r}_{3} = [M_{32}] (\vec{r}_{2} - \vec{r}_{1})
\] (11)

\[
\vec{r}_{3} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, \quad \vec{r}_{2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad [M_{32}] = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}
\]

Finish the formulas above,

\[
\begin{align*}
  x_{32} &= x_3 \cos \phi - y_3 \sin \phi + x_s \\
  y_{32} &= x_3 \sin \phi + y_3 \cos \phi + y_s
\end{align*}
\] (12)

The involute parameter equation is taken into the equation above to obtain the equation of the third involute segment in $(O_2-X_2Y_2)$.  

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\[
\begin{aligned}
&x_2 = -r \cos(v - \phi) - ru \sin(v - \phi) + x_i, \\
y_2 = r \sin(v - \phi) - ru \cos(v - \phi) + y_i,
\end{aligned}
\]  \quad (13)

Set \( v = v_1 + \Delta v (\Delta \tau \to \Theta) \) at the right end of the second involute segment and \( t = t_1 - \Delta t (\Delta \tau \to \Theta) \) at the left end of the third involute segment for ensuring it is unconditionally fourth order continuous and derivable. The calculation results are shown in Tab.2.

<table>
<thead>
<tr>
<th>Table 2. Continuous derivative calculation results at point C.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Left end point</strong></td>
</tr>
<tr>
<td>( \frac{dy_2}{dx_2} \mid_{x=r} = -\cot t_i )</td>
</tr>
<tr>
<td>( \frac{d^2 y_2}{dx_2^2} \mid_{x=r} = -\frac{1}{a t_i \sin^3 t_i} )</td>
</tr>
<tr>
<td>( \frac{d^3 y_2}{dx_2^3} \mid_{x=r} = \cos(v_i - \phi) + 3u_i \sin(v_i - \phi) )</td>
</tr>
</tbody>
</table>

Derivatives of left side and right side are equal to each other.

\[
x_2^* = x_i^*, y_2^* = y_i^* \quad (14)
\]

\[-\cot t_i = -\tan(v_i - \phi) \quad (15)\]

\[at_i \sin^3 t_i = rv_i \cos^3 (v_i - \phi) \quad (16)\]

\[
\frac{\sin t_i + 3t_i \cos t_i}{a^2 t_i^3 \sin^5 t_i} = \frac{\cos(v_i - \phi) + 3u_i \sin(v_i - \phi)}{r^3 v_i^3 \cos^3 (v_i - \phi)} \quad (17)
\]

Finish the formulas above.

\[
\begin{cases}
at_i = rv_i \\
t_i + v_i - \phi = 90^\circ
\end{cases} \quad (18)
\]

As long as the equations (14) and (18) are satisfied, it is fourth order continuous derivable at the connecting point \( C \), and the equation (18) is satisfied as long as the second involute segment and the third involute segment are center symmetry about the point \( C \).

The fourth involute segment and the third involute segment are symmetrical, and the construction method is the same as the second involute segment, so the profile is smooth. According to the curvature radius calculation formula and the Hertz contact formula, we can see that there is no abrupt change in curvature radius or Hertz stress at all connection points, so the method of constructing the tooth profile with four involute segments is reasonable and feasible.

**Ranges of Initial Construction Parameters**

The tooth height and tooth width of the high-close-order gear should be limited according to the standard of involute gear parameters, \( p = \pi m \), \( h = \frac{M_1}{m_1} \). As shown in Fig.4, the base circle is in \((O_1X_1Y_1)\), and \( AB \) presents the first two involute segments, whose base circle radius is \( a \) and generating angles are \( t_1 \) and \( t_2 \). Establish a new coordinate system \((O-X'Y')\), who has normal of profile for \( X \)-axis, tangential of profile for \( Y \)-axis, and point \( A \) is coordinate origin. The pressure angle is \( \alpha \), then \((Oa-XaYa)\) is obtained form rotation of \((O-X'Y')\) by \( \alpha \). \([Ma]\) and \([M']\) are rotation
matrices, and $\vec{r}_A$ is the translation vector. Addendum of high-close-order gear is $h$, and its pitch is $s$. Rotation angle is $\gamma = \pi - t$.

Then coordinate transformation equation between $(O_2-X_2Y_2)$ and $(O_1-X_1Y_1)$ can be obtained.

$$\vec{r}_a = [M_a][M'](\vec{r}_1 - \vec{r}_a)$$

$$\vec{r}_a = \begin{pmatrix} x_a \\ y_a \end{pmatrix}, \quad \vec{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \vec{r}_a = \begin{pmatrix} x_a \\ y_a \end{pmatrix}$$

$$[M_a] = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} [M'] = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix}$$

Bring the involute end point $B(x_B,y_B)$ into (19) to obtain the variation of gear height $h$ and pitch $s$ of the close gear with $t_1$ and $t_2$. The results are shown in Fig.5 and Fig.6.

Figure 4. Calculation of tooth width and addendum.

Figure 5. The Influence of $t_1$ and $t_2$ on addendum.

Figure 6. The Influence of $t_1$ and $t_2$ on pitch.
The involute generating angle is greater than zero, so $t_1$ is smaller than $t_2$, that means the correct value range should be on the left side of the curved surface. The horizontal section is the standard parameter of the involute gear. As shown in Fig.7, the curves obtained by the intersection of two curved surfaces represents the reasonable value range of $t_1$ and $t_2$. The radius of base circle has the effect of enlargement coefficient on pitch and addendum. Changing $a$ has no effect on the shape of the surface, but it directly acts on the maximal value on the surface, that means, the range of the value must ensure that the plane intersects the curved surface. So the ranges of the three parameters are determined according to constraint conditions $s_{\text{min}}(a) \leq s < s_{\text{max}}(a)$ and $h_{\text{min}}(a) \leq h < h_{\text{max}}(a)$.

**Gear-rack Meshing Calculation**

The meshing transmission of the rack and gear meets the tangential pure rolling on pitch. Based on the high-order-close rack profile of four involute segments above, the high-order-close gear profile equation can be calculated by tooth profile normal line method.

As is shown in Fig8, set equation of rack profile 1 as $f(x_1,y_1)$ in $(O_1-X_1Y_1)$. In order to make point $M$ the contact point, the tooth profile 1 should translate $l$ from the starting point, $l = x_1 + y_1 \tan(\gamma)$, and at the same time, the gear 2 rotates $\phi_2$ from the starting position. So the equation of meshing line can be calculated by flowing coordinate transformation.

$$\begin{align*}
  x &= x_1 - l \\
  y &= y_1
\end{align*}$$

The equation of gear profile 2 can be calculated by flowing coordinate transformation.
\[
\begin{align*}
\begin{cases}
    x_2 &= x_1 \cos \phi_2 + y_1 \sin \phi_2 + r_1 (\sin \phi_2 - \phi_2 \cos \phi_2) 
    \\
    y_2 &= -x_1 \sin \phi_2 + y_1 \cos \phi_2 + r_2 (\cos \phi_2 + \phi_2 \sin \phi_2)
  \end{cases}
\end{align*}
\]  

(21)

**Example**

Set modulus of high-order-close gear \( m = 2.5 \), tooth number \( z = 25 \), pressure angle \( \alpha = 20^\circ \), base circle radius \( r = 5 \text{mm} \) and initial angles \( \theta_1 = 30^\circ \), \( \theta_2 = 60^\circ \), and point \( C \) is the centrosymmetric point. The high-order-close gear profile is shown in Fig.9.

As shown in Fig.10, construct high-order-close gear profile drawing by Solidworks. As shown in Fig.11, plane model of high-order-close gear can be obtained according to the parameter standard of involute gear.
Conclusions

(1) The method of constructing tooth profile with four involute segments is feasible. Gear with great performances can be obtained as long as the parameters are selected properly. The high-order-close gear has better strength than traditional involute gear because of the concave-convex meshing form.

(2) The high-order-close gear profile is fourth-order continuous and derivable at all connecting points. According to the curvature radius calculation formula and the Hertz contact formula, it can see that there is no abrupt change in curvature radius or Hertz stress at all connecting points.

(3) The standardization process of this kind of gear and other performances need to be further studied.

References