Adaptive Synchronization of an Uncertain Complex Dynamical Network with Time-varying Delays

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Abstract. This paper investigates the locally and globally adaptive synchronization of an uncertain complex dynamical networks with time-varying delays. Some network synchronization criteria are proposed. The designed adaptive controllers for networks with unknown dynamics of nodes and unknown couplings are rather simple in form. It is very useful for future practical engineering design. Moreover, numerical simulations are demonstrated to illustrate the effectiveness of the proposed synchronization schemes.

Introduction

A complex network is a large set of interconnected nodes, where the nodes and connections can be anything. The structure of many real systems in nature can be described by complex networks, such as Internet, World Wide Web (WWW), communication networks, electrical power grids, social networks, and so on. Over the past decades, complex networks have attracted increasing attentions of researchers from various fields of science and engineering [1-6].

In recent years, the dynamics of complex networks has been extensively investigated. As a typical kind of dynamics, synchronization of all dynamical nodes in a complex network has received a great deal of interests. In fact, synchronization is a kind of typical collective behaviors and basic motions in nature [7-16]. It is a fundamental phenomenon that enables coherent behavior in networks as a result of interactions. For example, the synchronization of coupled oscillators can well explain many natural phenomena. Furthermore, some synchronization phenomena are very useful in our daily life, such as the synchronous transfer of digital or analog signals in communication networks.

Systems with time delays are quite ubiquitous in nature. In some network circumstances, delay couplings can be ignored on some links in the complex network, and others caused by traffic congestions or long distance between two nodes are significant, and must be considered. The time delays are usually caused by finite signal transmission speeds and memory effects. Due to the finite speeds of transmission and spreading as well as traffic congestions, a signal or influence traveling through a network is often associated with time delays, and this is very common in biological and physical networks. Therefore, time delays should be modeled in order to simulating more realistic networks. Some researchers have studied the synchronization of delayed complex networks [17-21]. In these investigations, an essential requirement is that the structure of the network and the coupling functions are known a priority. However, we often know very little information on the network structure, which makes network design very difficult. To overcome these difficulties, an effectively adaptive synchronization approach is proposed based on an uncertain complex dynamical network model with multiple time delays in this paper. Especially, our sufficient conditions for network synchronization are rather widely and the controllers are very simple. It is very useful for future practical engineering design.

The rest of this paper is organized as follows. Section 2 gives an uncertain complex network model and some useful preliminaries. In Section 3, locally and globally adaptive synchronization criteria for
uncertain complex networks with multiple time delays are proposed. In Section 4, a simulation example is provided to verify the effectiveness of the proposed method. Finally, conclusions are given in Section 5.

Preliminaries

This section introduces some mathematical definitions and hypotheses for an uncertain complex network model with time-varying delays. Consider the following complex network under control consisting of \( N \) identical nonlinear oscillators with uncertain nonlinear diffusive coupling, which is described by

\[
\dot{x}_i(t) = f(x_i(t)) + h_i(x_i(t-\tau(t)), x_2(t-\tau(t)), \cdots, x_N(t-\tau(t))) + u_i(t)
\]

(1)

where \( 1 \leq i \leq N \), \( x_i(t) = (x_{i1}, x_{i2}, \cdots, x_{im})^T \in \mathbb{R}^n \) is the state variable of the \( i \)th node at time \( t \), \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a smooth nonlinear vector field, the node dynamical function is \( \dot{x}(t) = f(x(t)) \), \( h_i: \mathbb{R}^n \rightarrow \mathbb{R}^n \) are unknown nonlinear smooth diffusive coupling functions, \( u_i(t) \) is the controller designed for the \( i \)th node, and the coupling control terms satisfy \( h_j(s(t),s(t),\cdots,s(t)) + u_i(t) = 0 \), where \( s(t) \) is a synchronous solution of node system \( \dot{s}(t) = f(s(t),t) \). Then \( S(t) = (s^T(t), s^T(t), \cdots, s^T(t))^T \) is a synchronous solution of complex network (1) since it is a diffusive coupling network. Here, \( s(t) \) can be an equilibrium point, a periodic orbit, an aperiodic orbit, or a chaotic orbit in the phase space.

Define error vector

\[
e_i(t) = x_i(t) - s(t)
\]

(2)

Then the objective of controller \( u_i(t) \) is to guide the complex network (1) to synchronize. That is,

\[
\lim_{t \rightarrow \infty} \|e_i(t)\| = 0, \quad 1 \leq i \leq N
\]

(3)

Since \( s(t) = f(s(t),t) \), from network (1), we have

\[
\dot{e}_i(t) = \tilde{f}(x_i(t), s(t), t) + \tilde{h}_i(x_i(t-\tau(t)), x_2(t-\tau(t)), \cdots, x_N(t-\tau(t)), s(t)) + u_i(t)
\]

(4)

where

\[
\tilde{f}(x_i(t), s(t), t) = f(x_i(t), t) - f(s(t), t), \quad \tilde{h}_i(x_i(t-\tau(t)), x_2(t-\tau(t)), \cdots, x_N(t-\tau(t)), s(t)) = h_i(x_i(t-\tau(t)), x_2(t-\tau(t)), \cdots, x_N(t-\tau(t))) - h_i(s(t), s(t), \cdots, s(t)), \quad 1 \leq i \leq N.
\]

In the following, we give several useful hypotheses.

Assumption 1. Assume that there exists a nonnegative constant \( \alpha \) satisfying

\[
\|Df(s(t),t)\| = \|A(t)\| \leq \alpha,
\]

where \( A(t) \) is the Jacobian of \( f(s(t),t) \) on \( s(t) \).

Assumption 2. Suppose that there exist a set of nonnegative constant \( \gamma_j \) (\( 1 \leq i, j \leq N \)), satisfying

\[
\|\tilde{h}_i(x_i(t-\tau(t)), x_2(t-\tau(t)), \cdots, x_N(t-\tau(t)), s(t))\| \leq \sum_{j=1}^{N} \gamma_j \|e_j(t-\tau(t))\|, \quad 1 \leq i \leq N.
\]

Assumption 3. Time delay \( \tau(t) \) is a differential function with \( 0 \leq \dot{\tau}(t) \leq \epsilon < 1 \). Clearly, this assumption is ensured if the coupling delay \( \tau(t) \) is a constant.

Remark 1. If Assumption 1 holds, then we can get

\[
\|A(t) + A^T(t)\| \leq \alpha.
\]

Lemma 1 [22]. For any vectors \( x, y \in \mathbb{R}^n \) and positive definite matrix \( Q \in \mathbb{R}^{n \times n} \), the following matrix inequality holds:

\[
2x^T y \leq x^T Q x + y^T Q y.
\]
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In this section, we will discuss the local synchronization and global synchronization of the uncertain complex networks with time-varying delays (1). Several network synchronization criterions are given.

Local Synchronization

Linearizing error system (4) at zero gives

\[ \dot{e}_i(t) = A(t)e_i(t) + \tilde{H}(x_1(t-\tau(t)), x_2(t-\tau(t)), \ldots, x_N(t-\tau(t)), s(t)) + u_i(t) \]

where \( 1 \leq i \leq N \) and recall that \( A(t) = Df(x, t) \) is the Jacobian of \( f \) evaluated at \( x = s(t) \).

Based on Assumption 1 and Assumption 2, a network synchronization criterion is deduced as follows.

**Theorem 1.** Suppose that Assumption 1, 2 and 3 hold. Then, the synchronous solution \( S(t) \) of uncertain complex network (1) is locally asymptotically stable under the adaptive controllers

\[ u_i = -k_i e_i, \quad 1 \leq i \leq N \]

and updating laws

\[ \dot{k}_i = \mu_i e_i^T e_i, \quad 1 \leq i \leq N \]

where \( \mu_i (1 \leq i \leq N) \) are positive constant.

**Proof:** Consider the Lyapunov candidate as follows:

\[ V = \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) e_i(t) + \frac{1}{2(1-\varepsilon)} \sum_{i=1}^{N} \frac{1}{\mu_i} (k_i - k^*)^2 + \frac{1}{2(1-\varepsilon)} \int_{t-\tau(t)}^{t} \sum_{i=1}^{N} e_i^T(\theta) e_i(\theta) d\theta \]

where \( k^* \) are positive constants to be determined. Thus one gets

\[ \dot{V} = \sum_{i=1}^{N} e_i^T(t) A e_i(t) + \sum_{i=1}^{N} \frac{k_i - k^*}{\mu_i} \dot{k}_i + \frac{1}{2(1-\varepsilon)} \sum_{i=1}^{N} e_i^T(t) e_i(t) - \frac{1}{2(1-\varepsilon)} \int_{t-\tau(t)}^{t} \sum_{i=1}^{N} e_i^T(\theta) e_i(\theta) d\theta \]

\[ = \sum_{i=1}^{N} e_i^T(t) A e_i(t) + \sum_{i=1}^{N} e_i^T(t) \tilde{H}(x_1(t-\tau(t)), x_2(t-\tau(t)), \ldots, x_N(t-\tau(t)), s(t)) - \sum_{i=1}^{N} \dot{k}_i e_i^T(t) e_i(t) \]

\[ = \sum_{i=1}^{N} e_i^T(t) \left( \frac{A - A^T}{2} e_i(t) + \sum_{i=1}^{N} e_i^T(t) \tilde{H}(x_1(t-\tau(t)), x_2(t-\tau(t)), \ldots, x_N(t-\tau(t)), s(t)) - \sum_{i=1}^{N} \dot{k}_i e_i^T(t) e_i(t) \right) \]

\[ = \sum_{i=1}^{N} e_i^T(t) \left( \frac{A - A^T}{2} e_i(t) + \sum_{i=1}^{N} \gamma_i e_i^T(t) e_i(t - \tau(t)) - \sum_{i=1}^{N} k_i e_i^T(t) e_i(t) \right) \]

\[ \leq \sum_{i=1}^{N} e_i^T(t) \left( \frac{A - A^T}{2} e_i(t) + \sum_{i=1}^{N} \gamma_i e_i^T(t) e_i(t - \tau(t)) - \sum_{i=1}^{N} k_i e_i^T(t) e_i(t) \right) \]

Let \( e(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T \in R^{2N} \), \( \Gamma = (\gamma_i)_{N \times N} \). Then by Lemma 1 and Assumption 3, we have

\[ \dot{V} \leq (\alpha - k^*) e_i^T(t) e_i(t) + e_i^T(t) \Gamma e(t - \tau(t)) - \frac{1}{2(1-\varepsilon)} e_i^T(t) e_i(t - \tau(t)) \]

where

\[ \alpha = \sum_{i=1}^{N} \frac{1}{\mu_i} (k_i - k^*)^2 \]

Let \( \alpha = k^* \) and \( \dot{k}_i = \mu_i e_i^T e_i \), then

\[ \dot{V} \leq (\alpha - k^*) e_i^T(t) e_i(t) + e_i^T(t) \Gamma e(t - \tau(t)) - \frac{1}{2(1-\varepsilon)} e_i^T(t) e_i(t - \tau(t)) \]

Thus the network (1) is locally asymptotically stable.
\begin{align*}
&\leq (\alpha - k^* + \frac{1}{2(1 - \epsilon)})e^T(t)e(t) + \frac{1}{2}e^T(t)\Gamma \Gamma e(t) + \frac{1}{2}e^T(t - \tau(t))e(t - \tau(t)) \\
&\quad - \frac{1 - \bar{\tau}(t)}{2(1 - \epsilon)}e^T(t - \tau(t))e(t - \tau(t)) \\
&\leq \left[ \alpha - k^* + \frac{1}{2(1 - \epsilon)} + \lambda_{\max} \frac{1}{2} \Gamma \Gamma \right] e^T(t)e(t)
\end{align*}

The constant $k^*$ can be properly chosen to make $\dot{V} \leq 0$. Therefore, based on the Lyapunov stability theory, the errors vector $e(t) \to 0$ as $t \to \infty$. That is, the synchronous solution $S(t)$ of uncertain complex network (1) is locally asymptotically stable under the adaptive controllers (6) and updating laws (7). The proof is thus completed.

**Global Synchronization**

In this section, the global network synchronization criterion is presented.

Rewrite node dynamics $\dot{x}_i(t) = f(x_i(t))$ as $\dot{x}_i(t) = Bx_i(t) + g(x_i(t))$, where $B \in \mathbb{R}^{n \times n}$ is a constant matrix, $g: \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear function. Thus complex network (1) is described by

\begin{equation}
\dot{x}_i(t) = Bx_i(t) + g(x_i(t)) + h_i(x_i(t - \tau(t)), x_{i_1}(t - \tau(t)), \ldots, x_{i_n}(t - \tau(t))) + u_i(t)
\end{equation}

where $1 \leq i \leq N$. Similarly, one can get the error system

\begin{equation}
\dot{e}_i(t) = Be_i(t) + \tilde{g}(x_i(t)) + \tilde{h}_i(x_i(t - \tau(t)), x_2(t - \tau(t)), \ldots, x_n(t - \tau(t)), s(t)) + u_i(t)
\end{equation}

where $1 \leq i \leq N$ and $\tilde{g}(x_i(t)) = g(x_i(t)) - g(s(t))$.

**Assumption 4.** Suppose there exists a constant $L > 0$, such that $\|g(x(t)) - g(y(t))\| \leq L\|x(t) - y(t)\|$ holds for any time-varying vectors $x(t), y(t)$, and the norm $\|\cdot\|$ of a vector $x(t)$ is defined as $\|x\| = (x^T x)^{1/2}$.

**Theorem 2.** Suppose that Assumption 2, 3 and 4 hold. Then, the synchronous solution $S(t)$ of uncertain complex network (1) is globally asymptotic stable under the adaptive controllers.

\begin{equation}
u_i = -k_i e_i, \quad 1 \leq i \leq N \tag{11}
\end{equation}

and updating laws

\begin{equation}k_i = \mu_i e_i^T e_i = \mu_i \|e_i\|^2, \quad 1 \leq i \leq N \tag{12}
\end{equation}

where $\mu_i (1 \leq i \leq N)$ are positive constant.

**Proof.** Since $B$ is a given constant matrix, there exists a nonnegative constant matrix, there exists a nonnegative constant $\beta$ such that $\|B\| \leq \beta$. It follows that $\|(B + B^T)/2\| \leq \beta$.

Similarly, constant Lyapunov function (8), then we can get

\begin{align*}
\dot{V} &= \sum_{i=1}^{N} e_i^T(t) \left( \frac{B + B^T}{2} \right) e_i(t) + \sum_{i=1}^{N} e_i^T(t) \tilde{g}(x_i(t)) + \sum_{i=1}^{N} e_i^T(t) \tilde{h}_i(x_i(t - \tau(t)), x_2(t - \tau(t)), \ldots, x_n(t - \tau(t)), s(t)) \\
&\quad - \sum_{i=1}^{N} k_i e_i^T(t)e_i(t) + \sum_{i=1}^{N} \frac{k_i}{\mu_i} \dot{k}_i + \frac{1}{2(1 - \epsilon)} \sum_{i=1}^{N} e_i^T(t)e_i(t) - \frac{1 - \bar{\tau}(t)}{2(1 - \epsilon)} \sum_{i=1}^{N} e_i^T(t - \tau(t))e_i(t - \tau(t)) \\
&\leq \sum_{i=1}^{N} e_i^T(t) \left( \frac{B + B^T}{2} \right) e_i(t) + \sum_{i=1}^{N} Le_i^T(t)e_i(t) + \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} e_i^T(t)e_j(t - \tau(t)) - \sum_{i=1}^{N} k_i e_i^T(t)e_i(t)
\end{align*}
\[
\begin{align*}
\dot{e}_i(t) &= \sum_{j=1}^{N} e_{ij}(t) - \frac{1 - \hat{e}(t)}{2(1-\varepsilon)} e_i(t - \tau(t)) e_i(t - \tau(t)) \\
&\leq (\beta - k^* + L + \frac{1}{2(1-\varepsilon)}) e(t) + e(t) \Gamma e(t - \tau(t)) - \frac{1 - \hat{e}(t)}{2(1-\varepsilon)} e(t - \tau(t)) e(t - \tau(t)) \\
&\leq (\beta - k^* + L + \frac{1}{2(1-\varepsilon)}) e(t) + \frac{1}{2} \dot{e}(t) \Gamma e(t) + \frac{1}{2} e(t - \tau(t)) e(t - \tau(t)) \\
&\leq \left[ \beta - k^* + L + \frac{1}{2(1-\varepsilon)} + \lambda_{\max} \left( \frac{1}{2} \Gamma \Gamma \right) \right] e(t)
\end{align*}
\]

The constant \( k^* \) can be properly chosen to make \( \dot{V} \leq 0 \). Therefore, based on the Lyapunov stability theory, the error vector \( e(t) \rightarrow 0 \) as \( t \rightarrow \infty \). That is, the global synchronous solution \( S(t) \) of uncertain complex network (1) is globally asymptotically stable under the adaptive controller (11) and updating laws (12). The proof is thus completed.

**Simulations**

In this section, a numerical example is presented to demonstrate the effectiveness of the proposed synchronization criteria. The following quantity

\[
E_j(t) = \sum_{i=1}^{N} e_{ij}^2(t) / N \quad j = 1, 2, \cdots, n
\]

is used to measure the quality of the synchronization process. It is obvious that when \( E_j(t) \) no longer increases, the drive and response networks achieve the desired synchronization globally.

Consider an example of controlled dynamical network (1), with the Lorenz system as a node. The number of the nodes is chosen as \( N = 50 \). Here, a single Lorenz system of node \( i \) is described by \[23\]

\[
\begin{pmatrix}
\dot{x}_{i1} \\
\dot{x}_{i2} \\
\dot{x}_{i3}
\end{pmatrix} = \begin{pmatrix}
x_{i1} \\
x_{i2} \\
x_{i3}
\end{pmatrix} + \begin{pmatrix}
0 \\
-x_{i1} x_{i3} \\
x_{i1} x_{i2}
\end{pmatrix}
\]

(13)

Where, \( A = \begin{pmatrix}
-a & a & 0 \\
c & -1 & 0 \\
0 & 0 & -b
\end{pmatrix} \), \( a = 10 \), \( b = 8/3 \), \( c = 28 \), and \( 1 \leq i \leq 50 \). The chaotic attractor of Lorenz system is shown in Fig. 1.

Figure 1. The Lorenz attractor.
The networked system is defined as follows:

\[
\begin{align*}
\dot{x}_{i1} &= A \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} + \begin{pmatrix} f_1(x_i) - 2f_i(x_{i1}) + f_i(x_{i+2}) \\ f_2(x_i) - 2f_i(x_{i1}) + f_i(x_{i+2}) \end{pmatrix} - k_ie_i
\end{align*}
\]

and

\[
\dot{k}_i = \mu \dot{e}_i^T e_i = \mu_i \|e_i\|^2
\]

\[
f_1(x_i) = a(x_{i2} - x_{i1}), \quad f_2(x_i) = x_{i1}x_{i2} - bx_{i3}, \quad x_{i1} \equiv x_1, \quad x_{i2} \equiv x_2, \quad \text{and } 1 \leq i \leq 50
\]

Obviously, one gets

\[
\bar{g}(x_i, s) = \begin{pmatrix} 0 \\ -x_{i1}x_{i3} + s_i s_3 \\ x_{i1}x_{i2} - s_i s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_{i3} e_{i1} - s_i e_{i3} \\ x_{i2} e_{i1} + s_i e_{i2} \end{pmatrix}
\]

where \(1 \leq i \leq 50\).

Since Lorenz attractor is bounded in a bounded region \(\Phi \subset \mathbb{R}^3\), there exists a constant \(M\) satisfying

\[
|e_i|, |s_i| \leq M \quad \text{for } 1 \leq i \leq 50 \quad \text{and} \quad j = 1, 2, 3.
\]

Therefore,

\[
\|g(x_i, s, t)\|_2 = \sqrt{(x_{i3} e_{i1} + s_i e_{i3})^2 + (x_{i2} e_{i1} + s_i e_{i2})^2} \leq 2M \|e_i\|_2
\]

Similar to [24, 25], one has

\[
\bar{h}(x_i(t - \tau(t)), x_{i1}(t - \tau(t)), \ldots, x_N(t - \tau(t))) \leq 3\sqrt{2(a^2 + M^2)}(\|e_i(t - \tau(t))\|_2 + \|e_{i+1}(t - \tau(t))\|_2 + \|e_{i+2}(t - \tau(t))\|_2)
\]

Thus Assumption 2 and Assumption 4 hold. According to Theorem 2, the synchronous solution \(S(t)\) of dynamical network (14) is globally asymptotic stable.

Assume that \(\mu_i = 1, \quad k_i(0) = 1, \quad x_i(0) = (4 + 0.5i, 5 + 0.5i, 6 + 0.5i)\) for \(1 \leq i \leq 50\) and \(\dot{x}(0) = (4, 5, 6)\).

The synchronous error \(E_j(t)\) is shown in Fig. 2. Obviously, the zero is globally asymptotic stable for dynamical network (14).

\[\text{Figure 2. Synchronization errors of the network with time-varying delays.}\]

**Conclusions**

In this paper, we have investigated the locally and globally adaptive synchronization of an uncertain complex dynamical network with time-varying delays. Several novel network synchronization criteria have been proved based on the Lyapunov stability theory. The designed linear feedback controllers are simple. It is useful for future practical engineering design. Furthermore, the
effectiveness of the proposed synchronization criteria are demonstrated by numerical simulations.

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