The Anti-periodic Solutions Set for a Class of Nonlinear Evolution Inclusions in $R^N$

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Abstract. We study the structural properties of the anti-periodic solutions set for a class of nonlinear evolution inclusions in $R_δ$. When the right-hand side term is convex-valued, we obtain that the anti-periodic solutions set is a compact $R_δ$ set.

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Introduction

In this paper, we examine the structural properties of the anti-periodic solution set for a class of nonlinear evolution inclusions based on [1,2] in $R_δ$. The anti-periodic problems of evolution inclusions were investigated by Okochi [3], Aizicovici-McKibben-Reich [4], Franco-Nieto-O'Regan [5], Chen-Cho-O'Regan [6], Park-Ha [7], and Liu [8] and the references therein. In the past the topological structure of the solution set of differential inclusions in $R_δ$ has been investigated by Himmelberg-Van Vleck [9] and DeBlasi-Myjak [10]. Himmelberg and Van Vleck considered the topological structure of the solution set to the following differential inclusions

$$\dot{x}(t) \in F(t,x(t)),$$

and obtained that the solution set was an $R_δ$-set. For the Cauchy problems the topological structure of the solution set of evolution inclusions was examined primarily by Papageorgiou-Shahzad [11] and Papageorgiou-Yannakakis [12] in a Banach space. For the optimization of this subject, we refer the reader to the work of [21]. However, none of these works addressed the topological structure of the anti-periodic solution set studied in this paper. In this paper we prove that the solution set of nonlinear time-dependent evolution inclusions with a convex-valued orientor field is compact $R_δ$ in $C(I,R^N)$.

Preliminaries

We still use the notation introduced in [1,2]. Let $(\Omega,\Sigma)$ be a measurable space and $X$ a separable Banach space. Throughout this note, we use the following notation:

$$P_{f(0)}(X) = \{ A \subseteq X : nonempty, closed, (and convex) \}.$$ A multifunction $F: \Omega \to P_f(X)$ is said to be measurable if, for all $x \in X$, the $R_δ$-valued function $\omega \to d(x,F(\omega))$ is measurable. Let $Y,Z$ be Hausdorff topological spaces and $G:Y \to 2^Z / \{ \emptyset \}$. We
say that $G(\cdot)$ is upper semicontinuous (USC) (resp. lower semicontinuous (LSC)) if, for all $U \subseteq Z$ nonempty, open, $G^+ (U) = \{ y \in Y : (G(y) \subseteq U) \}$ (resp. $G^- (U) = \{ y \in Y : (G(y) \cap U \neq \emptyset) \}$) is open in $Y$.

On $P_f (X)$ we can define a generalized metric, known in the literature as the Hausdorff metric, by

$$h(A, B) = \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \}.$$  

The metric space $(P_f (X), h)$ is complete and a multifunction $G : X \rightarrow P_f (X)$ is said to be Hausdorff continuous ($h$-continuous) if it is continuous from $X$ into $(P_f (X), h)$.

**Proposition 2.1** If $F : I \times H \rightarrow P_{S^+} (V^*)$ is measurable in $t$, $h$-USC in $x$, and $|F(t, x)| \leq \varphi(t)$ a.e. with $\varphi(t) \in L^p (I)$, then there exists a sequence of multifunctions $F_n : I \times H \rightarrow P_{S^+} (V^*)$, $n \geq 1$, such that for every $x \in H$ there exist $\mu(t) > 0$ and $\epsilon > 0$ such that if $x_1, x_2 \in B_\varepsilon (x) = \{ y \in H : \| x - y \| \leq \varepsilon \}$, then

$$h(F_n (t, x_1), F_n (t, x_2)) \leq \mu(t) \varphi(t) \| x_1 - x_2 \| \text{ a.e. (i.e., } F_n (t, x) \text{ is locally } h\text{-Lipschitz),}$$

$F(t, x) \subseteq \cdots \subseteq F_n (t, x) \subseteq \cdots \subseteq F_{n+1} (t, x) \cdots$, $F_n (t, x) \leq \varphi(t)$ a.e. $n \geq 1$, $F_n (t, x) \rightarrow F(t, x)$ as $n \rightarrow 1$ for every $(t, x) \in I \times H$, and there exists $u_n : I \times H \rightarrow H$, measurable in $t$, locally Lipschitz in $x$ (as $F_n (t, x)$) and $u_n (t, x) \in F_n (t, x)$ for every $(t, x) \in I \times H$. Moreover, if $F(t, \cdot)$ is $h$-continuous, then $t \rightarrow F_n (t, x)$ is measurable (hence $t \rightarrow F_n (t, x)$ is measurable too; see [14]).

**Main Results**

Let $I = [0, T]$, consider the following anti-periodic problems of evolution inclusions

\begin{align*}
\dot{x} + A(t, x) &\in F(t, x), \text{ a.e. } I \\
x(0) &= -x(T).
\end{align*}

We denote the solution set of (1) by $S$, and will show that $S$ is an $R_\delta$ set in $C(I, R^N)$. To this end we need the following hypotheses on the data of (1).

(H1) $A : I \times R^N \rightarrow R^N$ is an operator such that (i) $t \rightarrow A(t, x)$ is measurable; (ii) for each $t \in I$, the operator $A(t, \cdot) : R^N \rightarrow R^N$ is uniformly monotone and hemicontinuous, that is, there exists a constant $p > 0$ such that

$$\langle A(t, x_1) - A(t, x_2), x_1 - x_2 \rangle \geq p \| x_1 - x_2 \|$$

for all $x_1, x_2 \in R^N$, and the map $s \mapsto \langle A(t, x + sz), y \rangle$ is continuous on $[0, 1]$ for all $x, y, z \in R^N$.

For every $f \in L^2 (I, R^N)$, the following nonlinear evolution equation

\begin{align*}
\dot{x} + A(t, x) &= f(t), \text{ a.e. } I \\
x(0) &= -x(T)
\end{align*}

has a unique solution $x = P (f) \in C(I, R^N)$, where $P$ is defined as the solution map of problem (2).

**Proposition 3.1** If hypotheses (H1) hold, then $P : L^2 (I, R^N) \rightarrow C(I, R^N)$ is sequentially continuous.

**Proof.** As in [2], let

$$W = \{ v \in L^2 (I, R^N) : \| v \| \leq \psi(t) \text{ a.e. on } I \},$$

where
with $\psi(t) \in L^2(I)$, then $K = \{P(v) : v \in W\} \subseteq W^{1,2}(I, R^N)$ is compact convex subset in $C(I, R^N)$. Let $\{f_n\}_{n \geq 1} \subseteq W$. From the definition of $W$, so $W$ is uniformly bounded in $L^2(I, R^N)$. By Dunford-Pettis theorem, passing to a subsequence if necessary, we may assume that $f_n \to f$ weakly in $L^2(I, R^N)$ for some $f \in W$. Set $x_n(t) = P(f_n)$ and $x(t) = P(f)$. From a prior estimation of solution in [1], we have $x_n(t) \to x(t)$ in $L^2(I, R^N)$, so $x_n(0) \to x(0)$ as $n \to \infty$. Therefore, we have

$$
\|x_n(t) - x(t)\| \leq 2 \int_0^T (f_n(s) - f(s), x_n(s) - x(s)) ds + \|x_n(0) - x(0)\|
$$

$$
\leq 4 \int_0^T \|\psi\|_{C^0} x_n - x\|dt + \|x_n(0) - x(0)\|
$$

$$
\leq 4 \|\psi\|_{C^0} \|x_n - x\|_2 + \|x_n(0) - x(0)\|
$$

$$
\to 0 \text{ as } n \to \infty.
$$

We see that

$$
\max_{t \in I} \|x_n(t) - x(t)\| \to 0 \text{ as } n \to \infty.
$$

So, $x_n(t) \to x(t)$ in $C(I, R^N)$, i.e. $P: L^2(I, R^N) \to C(I, R^N)$ is sequentially continuous.

To obtain such a structural result for the solution set of (0.1) we need the following hypotheses on $F(t, x)$:

(H2) $F: I \times R^N \to P_{R^N}(R^N)$ is a multifunction such that (i) $(t, x) \to F(t, x)$ is graph measurable; (ii) for almost all $t \in I$, $x \to F(t, x)$ has a closed graph; (iii) there exists an nonnegative function $b(\cdot) \in L^1(I)$ and a constant $C > 0$ such that

$$
|F(t, x)| = \sup\{\|f\| : f \in F(t, x)\} \leq b(t) + C\|x\|^{\alpha}, \forall x \in R^N \text{ a.e. } I
$$

where $0 \leq \alpha \leq 1$.

**Theorem 3.1** If hypotheses (H1) and (H2) hold, then $S$ is an $R$ set in $C(I, R^N)$.

**Proof.** From the a prior estimation conducted in the proof of Theorem 3.2 in [1], we know that without loss of generality, we may assume that for almost all $t \in I$, all $x \in R^N$ and all $v \in F(t, x)$, we have $\|v\| \leq \psi(t)$ with $\psi \in L^1(I)$. Apply Proposition 2.1 to generate a sequence of multifunctions $F_n: I \times R^N \to P_{R^N}(R^N)$. For every $n \geq 1$, consider the following anti-periodic problem of evolution inclusion:

$$
x + A(t, x) \in F_n(t, x), \text{ a.e. } I
$$

$$
x(0) = -x(T).
$$

From Theorem 3.3 of [1], we obtain that problem (4) has a nonempty solution set

$$
S_n \subseteq W^{1,2}(I, R^N) \cap C(I, R^N)
$$

which is compact in $C(I, R^N)$. The rest part of the proof is divided into two steps.

**Step 1.** We will claim that this set $S_n$ is contractible. Let $f_n(t, x)$ be the locally Lipschitz with respect to $x \in R^N$, measurable selector of $F_n(t, x)$ postulated by Proposition 2.1. Let $\gamma = [0, 1]$, for each $\rho \in \gamma$, let $u_{\rho}$ denote the unique solution of the following equation

$$
\dot{x} + A(t, x) \in f_n(t, x), \text{ a.e. } I
$$

$$
x(0) = -x(T).
$$
\[ \dot{u}(t) + A(t,u(t)) = f_n(t,u), \quad \text{a.e. on } [\rho T, T], \]
\[ u(\rho T) = -x(\rho T). \]
for given \( x \in S_n \). Then we can define a function \( \eta(\rho, x) : \gamma \times S_n \to S_n \) by

\[ \eta(\rho, x)(t) = \begin{cases} x(t) & \text{for } t \in [0, \rho T], \\ u(\rho, x)(t) & \text{for } t \in [\rho T, T]. \end{cases} \]

Evidently, for every \( x \in S_n \), if \( \rho = 0 \), then \( \eta(0, x)(0) = x(0) \), and if \( \rho = 1 \), then \( \eta(1, x)(t) = x(t) \). If we can show that \( \eta(\rho, x) \) is continuous in \( C(I, R^N) \), then we will show that \( S_n \) is contractible in \( C(I, R^N) \). To this end let \((\rho_m, x_m) \to (\rho, x) \) in \( \gamma \times S_n \). We consider one cases, the other is similar. Without loss of generality, let \( \rho_m \geq \rho \) for every \( m \geq 1 \). Set \( v_m(t) = \eta(\rho_m, x_m)(t) \), for each \( t \in I \).

Evidently \( v_m(t) \in S_n \), for every \( m \geq 1 \). From the compactness of \( S_n \) in \( C(I, R^N) \), and so by passing to a subsequence if necessary, we may assume that \( v_m \to v \) in \( C(I, R^N) \) as \( m \to \infty \). Clearly \( v(t) = x(t) \) for \( 0 \leq t \leq \rho T \). Also let \( z \in W^{1,2}(I, R^N) \) be the unique solution of

\[ \dot{z} + A(t, z) = f_n(t, v), \quad \text{a.e. on } [\rho T, T], \]
\[ z(\rho T) = -v(\rho T). \]

Let \( M \geq 1 \), then for \( m \geq M \) large enough \( v_m(t) \) satisfies \( \dot{v}_m + A(t, v_m) = f_n(t, v_m) \) a.e. on \([\rho_M T, T]\). Because of the uniformly boundness of \( f_n \), by passing to a subsequence if necessary we may assume that \( \dot{v}_m \to \dot{v} \) in \( W^{1,2}(I, L^2) \). Then \( z(\rho T) \to z(\rho T) \) in \( R^N \) as \( M \to \infty \), in the limit we have \( z(t) = v(t) \) for every \( t \in [\rho T, T] \). From (7), we have

\[ \dot{z} + A(t, z(t)) = f_n(t, z(t)), \quad \text{a.e. on } \rho T \leq t \leq T, \]
\[ z(\rho T) = -x(\rho T). \]

hence \( v = \eta(\rho, x) \). Therefore, \( \eta(\rho_m, x_m) \to \eta(\rho, x) \) in \( C(I, R^N) \).

In general we can always find a subsequence of \( \{\rho_m\}_{m \geq 1} \) satisfying \( \rho_m \leq \rho \) or \( \rho_m \geq \rho \). So we have proved the continuity of \( \eta(\rho, x) \). Therefore, for every \( n \geq 1 \), \( S_n \subseteq C(I, R^N) \) is compact and contractible.

**Step 2.** We will claim that \( S = \bigcap_{n \geq 1} S_n \). Clearly, \( S \subseteq \bigcap_{n \geq 1} S_n \). Let \( x \in \bigcap_{n \geq 1} S_n \). Then by definition \( x = P(f_n) \), \( f_n \in S^2_{F(x_n)} \) for some \( n \geq 1 \), where \( S^2_{F(x)} \) denotes the set of all \( L^2(I, R^N) \)-selection of \( F_x \). Because of the uniformly boundness of \( f_n \), by passing to a subsequence if necessary we may assume that \( f_n \to f \) weakly in \( L^2(I, R^N) \). Then \( f \in S^2_{F(x_n)} \) (see Theorem 3.2 of [1]). So \( x = P(f) \) with
\[ f \in S^2_{f(x,t)} \] from which we conclude that \( \bigcap_{n=1} S_n \subseteq S \), i.e. \( \bigcap_{n=1} S_n = S \). Finally, Hyman's result [12] implies that \( S \) is a compact \( R_\delta \) set in \( C(I, \mathbb{R}^N) \).

We know the contractible set is connected, so an immediate consequence of Theorem 3.1 is the following result for the multivalued problem (1).

**Remark 3.1** If hypotheses (H1)-(H2) hold, then for every \( t \in I \), \( S(t) = \{ x(t) | x \in S \} \) (the reachable set at time \( t \in I \)) is compact and connected in \( \mathbb{R}^N \).

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**References**


