A Mathematical Approach to Study the Blood Flow Through Tapered Stenosed Artery with the Suspension of Nanoparticles

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Abstract. This model of heat and mass transfer on the blood flow through a tapered stenosed artery is solved by Homotopy Perturbation Method where the rheology of flowing blood is characterized by the Jeffrey fluid model. The equations governing the blood flow is modeled in cylindrical coordinates. Analytical solutions are constructed for the velocity, temperature, concentration and flux by solving flow governing non-linear coupled equations. Variation in velocity profile, temperature profile, concentration profile and flux profile for different values of thermophoresis and Brownian motion parameter are discussed.

Introduction

According to the Global Burden of Disease Study(2013), 17.3 million deaths across the world in 2013 were related to cardiovascular diseases[10,12]. There was an increment of 41% since 1990. Undoubtedly, cardiovascular diseases are the leading cause of morbidity and mortality of our generation, more than 30% of all deaths in people aged 35 and above[9]. Atherosclerosis is starting at a young age, with progressive plaque deposition in the major arteries of the body. When such a plaque becomes big enough in the coronary artery, myocardial ischemia or infarction can follow[1,4]. Since the pathogenesis of atherosclerosis starts at the cellular level, only an effective intervention at this level can thwart its progression. Atherosclerosis is a chronic inflammatory disease of the arterial wall that arises from an imbalanced lipid metabolism and a maladaptive inflammatory response [3,5].

![Figure 1. Blood flow in a stenosed tapered artery.](image)

Despite intensive research on mechanisms underlying atherosclerotic lesion formation and progression during the past decade, translation of this knowledge into the clinic is scarce. Nanotechnology holds tremendous potential to advance the current treatment of coronary artery disease [6,11]. Nanotechnology may assist medical therapies by providing a safe and efficacious delivery platform for a variety of drugs aimed at modulating lipid disorders, decreasing inflammation and angiogenesis within atherosclerotic plaques, and preventing plaque thrombosis. Nanotechnology may improve coronary stent applications by promoting endothelial recovery on a
Problem Statement and Mathematical Formulation

Let us consider one dimensional pulsatile, axially symmetric, laminar, incompressible, fully developed flow of blood is treated as Jeffrey fluid with nano-particles, having constant viscosity $\mu$ and density $\rho$, through a tube shaped artery of radius $R_0$ and length $L$ [12]. The geometry of the arterial wall with overlapping stenosis [Figure 1] is given as [5]:

$$\psi(z) = 1, \text{ otherwise}$$

$$\psi = \frac{(\delta) n^{n-1}}{R_0 L_0^{n-1}(n-1)}$$

$$d(z) = R_0 + \xi z,$$

in which $\delta$ denotes the maximum height of the stenosis located at

$$z = d_0 + \frac{L_0}{n^{n-1}}$$

The equations governing the flow are:

$$\frac{v}{r} + \frac{\partial(v)}{\partial r} + \frac{\partial u}{\partial z} = 0$$

$$\rho \left( \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( r S_{rr} \right) + \frac{\partial}{\partial z} \left( S_{rz} \right) - \frac{1}{r} \left( S_{r\theta} \right),$$

$$\rho \left( \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r S_{rz} \right) + \frac{\partial}{\partial z} \left( S_{zz} \right) + \rho \alpha_1 (T - T_1) + \rho \alpha_1 (C - C_1)$$

$$\left( \frac{\partial T}{\partial r} + u \frac{\partial T}{\partial z} \right) = \alpha_1 \left( \frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial z^2} \right) + \frac{D_T}{T_0} \left( \frac{\partial T}{\partial r} \right)^2 + \left( \frac{\partial T}{\partial z} \right)^2$$

$$\left( \frac{\partial C}{\partial r} + u \frac{\partial C}{\partial z} \right) = D_B \left( \frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial z^2} \right) + \frac{D_T}{T_0} \left( \frac{\partial T}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}$$

$$S_{rr} = \frac{2\mu}{1 + \lambda_1} \left( 1 + \lambda_2 \left( \frac{v}{\frac{\partial r}{\partial r} + u \frac{\partial z}{\partial z}} \right) \frac{\partial v}{\partial r} \right),$$

$$S_{rr} = \frac{\mu}{1 + \lambda_1} \left( 1 + \lambda_2 \left( \frac{v}{\frac{\partial r}{\partial r} + u \frac{\partial z}{\partial z}} \right) \left( \frac{\partial v}{\partial z} + \frac{\partial u}{\partial r} \right) \right),$$

$$S_{zz} = \frac{2\mu}{1 + \lambda_1} \left( 1 + \lambda_2 \left( \frac{v}{\frac{\partial r}{\partial r} + u \frac{\partial z}{\partial z}} \right) \frac{\partial u}{\partial z} \right),$$

Where $\lambda_1$ is the ratio between relaxation to retardation times, and $\lambda_2$ is the retardation time.

Defining:

$$r' = \frac{r}{R_0}; \quad z' = \frac{z}{L_0}; \quad v' = \frac{L_0}{\frac{\partial r}{\partial r}}; \quad u' = \frac{u}{\frac{\partial z}{\partial z}};$$

$$R' = \frac{R}{R_0}; \quad P' = \frac{P}{\frac{\partial r}{\partial r}}; \quad C' = \frac{C - C_1}{C_0 - C_1};$$
\begin{equation}
G_r = \frac{\rho g \alpha_1 R_0^3}{\mu} (T_0 - T_1); \quad B_r = \frac{\rho g \alpha_1 R_0^3}{\mu} (C_0 - C_1); \quad R_e = \frac{\rho U R_0}{\mu};
\end{equation}

\begin{align*}
N_t &= \frac{(\rho c)_p D_r T_0}{(\rho c)\alpha_3}, \quad N_b = \frac{(\rho c)_p D_B C_0}{(\rho c)\alpha_3};
\end{align*}

(10)

Where \( R_e \) is the Reynolds number, \( N_t \) is the thermophoresis parameter, \( N_b \) is the Brownian motion parameter, \( G_r \) is the local temperature Grashof number, \( B_r \) is the local Grashof number.

Using the non-dimensional variables in equation (10) along with the additional boundary conditions.

1). \( \frac{R_e \delta n^{n-1}}{L_o} \ll 1 \), \( \delta^* = \frac{R_e \delta n^{n-1}}{L_o} \sim o(1) \),

And for mild stenosis \( \left( \frac{\delta}{R_0} \ll 1 \right) \) in equation (5) to (9), after dropping the dashes take the form

\begin{align*}
\delta^* \left( \frac{\partial r}{\partial r} + \frac{v}{r} \right) + \frac{\partial u}{\partial z} &= 0, \\
\frac{\partial p}{\partial r} &= 0, \\
\frac{\partial p}{\partial z} &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \left( \frac{\partial u}{\partial r} \right) \right) + G_r \varphi + B_r \sigma, \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r \left( \frac{\partial \varphi}{\partial r} \right) \right) + N_d \frac{\partial \varphi}{\partial r} + N_t \left( \frac{\partial \varphi}{\partial r} \right)^2 &= 0, \\
N_b \frac{\partial}{\partial r} \left( r \left( \frac{\partial \varphi}{\partial r} \right) \right) + N_t \frac{\partial}{\partial r} \left( r \left( \frac{\partial \varphi}{\partial r} \right) \right) &= 0
\end{align*}

(11) (12) (13) (14) (15)

After integration equation (15) gives the result

\[ \sigma = -\varphi \frac{N_t}{N_b} \]

(16)

The boundary conditions are as follows:

\begin{align*}
\frac{\partial u}{\partial r} &= 0, \quad \frac{\partial \varphi}{\partial r} = 0, \quad \frac{\partial \varphi}{\partial r} = 0 \quad \text{at } r = 0 \\
w &= 0, \quad \varphi = 0, \quad \sigma = 0 \quad \text{at } r = R(z), \\
R(Z) &\frac{1+\xi_1 z}{\xi_1 z} = [1 - \psi_1 ((z - d_0) - (z - d_0^*)^n)], \quad d_0^* < z \leq d_0 + 1,
\end{align*}

(17)

\[ d_0^* = \frac{d_0}{L_0}, \quad \xi_1 = \frac{\xi L_0}{R_0}, \quad \psi_1 = \delta^* \frac{n^{n-1}}{(n-1)}. \]

Solution of the Problem Using Numerical and Analytical Applied Methods

The solution of the equation (14) are calculated by homotopy perturbation method as

\[ H(k, \varphi) = (1 - k)[L(\varphi) - L(\varphi_{10})] + k \left[ L(\varphi) + N_d \frac{\partial \varphi}{\partial r} + N_t \left( \frac{\partial \varphi}{\partial r} \right)^2 \right], \]

(18)

Which has the range \( 0 \leq k \leq 1 \), \( L = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) \), is a linear operator.

\[ \varphi_{10}(r, z) = -\left( \frac{r^2 - R^2}{4} \right) \]

(19)

\[ \varphi = \varphi_0 + k \varphi_1 + k^2 \varphi_2 + o(k^3) \]

(20)
Putting equations (20) in equation (14), and taking $k \to 1$, the following expression for temperature profile is obtained as follows:

$$\varphi(r,z) = (2N_t + N_b) \left( \frac{r^4 - R^4}{64} \right) - \left( \frac{r^6 - R^6}{1152} \right) (2N_t + N_b)(N_t + N_b).$$

(21)

Using the above result of temperature profile in equation (16), we get:

$$\sigma(r,z) = \frac{N_t}{N_b} \left(2N_t + N_b\right) \left( \frac{r^4 - R^4}{64} \right) - \left( \frac{r^6 - R^6}{1152} \right) (2N_t + N_b)(N_t + N_b).$$

(22)

By putting equation (21) and (22) in equation (13) we get the result for velocity profile as

$$u(r,z) = \frac{r^2}{2} \left(1 + \lambda_1\right) \frac{d\varphi}{dz} - \left(G_r - B_r \frac{N_t}{N_b}\right) (2N_t + N_b)(1 + \lambda_1) \left(\frac{r^7 - 21r^5R^4}{8064} - \frac{r^9 - 12r^3R^6}{82944}\right).$$

(23)

**Results and Discussions**

In the present study, the nature of blood in arteries as non-Newtonian fluid is investigated analytically. Homotopy perturbation method is applied to solve the temperature profile governing equation. The result of temperature profile is used to evaluate results for concentration and velocity profile. In order to have estimate of the quantitative effects of various parameters involved in the analysis computer codes were developed and to evaluate the analytical results obtained for temperature, concentration and velocity profile.

Where figure 2 depicts the variation of velocity profile $u(r,z)$ with radius of artery with stenosis $R(z)$ for different values of Grashof number $G_r$. It is observed in the figure that with increase in Grashof number $G_r$ the velocity profile $u(r,z)$ increases[7,12]. But interestingly it is found that at
R(z) = 0, the velocity profiles varies in different manner, i.e. with increase in Grashof number the velocity profile decreases. Where figure 3 depicts the variation of velocity profile $u(r,z)$ with radius of artery with stenosis R(z) for different values of local Grashof number $B_r$. It is observed in the figure that with increase in local Grashof number $B_r$ the velocity profile $u(r,z)$ decreases. But interestingly it is found that at R(z) = 0, the velocity profiles varies in different manner, i.e. with increase in local Grashof number $B_r$ the velocity profile increases[5].

Figure 4 shows the variation of temperature profile for different values of Brownian motion parameter. It is observed from the figure that with increases in Brownian motion parameter temperature profile decreases. Figure 5 shows the variation of temperature profile $\varphi(r,z)$ for different values of thermophoresis parameter $N_t$. It is observed from the figure that with increase in thermophoresis parameter $N_t$ temperature profile $\varphi(r,z)$ decreases[3,7].

**Conclusion**

Metallic nanoparticles analysis through an axisymmetric mild stenosis where blood is considered as non-Newtonian Jeffrey fluid have been solved by using a novel method called Homotopy perturbation method. The heat and mass transfer via nanoparticles are also taken into account. It seen in this present analysis that the velocity profile decreases with an increase in the Grashof number and local Grashof number and the temperature profile decreases with an increase in Brownian motion parameter as well as thermophoresis parameter. It is also seen in this analysis that the concentration profile decreases with an increase in the Brownian motion parameter and thermophoresis parameter.

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**References**


