Random Response of Functionally Graded Material Beam with Viscoelastic Damping

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Abstract. The random response of the functionally graded material beam with viscoelastic damping under Gaussian white noise is investigated by an approximate transformation procedure and equivalent non-linearization technique. Firstly, the dimensionless equation of the functionally graded material beam is obtained by the Galerkin method, and the viscoelastic force is approximated to the equivalent damping and the equivalent stiffness, which depend on the system averaged frequency. Secondly, the equivalent non-linearization technique is used, and the random response of the original system is obtained. Finally, the influences of the different parameters on the system response are discussed, especially the viscoelastic parameter $\beta_1$ and $\lambda_1$.

Introduction

Functionally graded materials [1] are composite materials in which the physical and mechanical properties varying spatially along specific directions. In the thermal environment, Metal/ceramic and polymer/ceramic functionally graded materials exhibit time-dependent mechanical properties-stress relaxation and creep properties [2]. Therefore, it is necessary to study the viscoelastic behavior of functionally graded materials. At present, there are a large number of literatures on the study of functionally graded viscoelastic materials, and some analytical methods and control strategies have been developed. Based on the Norton slaw, Chen and You [3, 4] assumed that the creep coefficient of the gradient material was a function of the radial coordinate, and the spherical symmetric deformation solution of spherical vessel of functionally graded material and the ax symmetric deformation solution of the thick-walled cylindrical vessels was obtained, respectively. Some scholars utilized the viscoelastic mechanics model to describe the heat flux of functionally graded materials. Mukherjee and Khazanovich [5, 6] proposed the elastic-viscoelastic correspondence principle. Zhang and Wang [7] analyzed the thermo viscoelastic deformations of functionally graded thin plates. Li et al. [8] used the Paulino’s correspondence principle [5] to study the viscoelastic fracture of functionally graded material strip with a crack normal to the gradient direction, and obtained the stress intensity factor of the viscoelastic functionally graded material strip.

Considering the viscoelastic behavior in the functionally gradient materials, this manuscript aims to investigate the random vibration of the functionally graded material beam with the viscoelastic damping. At present, the stochastic averaging technique has attracted much attention in many methods of studying the nonlinear stochastic system response. However, due to its limited resistance to weak damping and weak excitation intensity, it cannot be extended to large damping and strong excitation, and the deriving...
process is difficult to understand by most people. On the contrary, the equivalent on-linearization technique has the advantages of analyzing the large damping and strong excitation conditions, the analysis process is simple and easy to understand, and thus has a greater advantage. Zhu et al. [9] have developed the equivalent on-linearization technique under the Hamiltonian framework, and proposed three kinds of equivalent criterion: the least mean-squared deficiency of damping forces, dissipation energy balancing and least mean-squared deficiency of dissipation energies. Recently, the equivalent non-linearization technique has been applied to predict the random response of the vibro-impact system [10]. The results show that the method is still highly accurate for the strong excitation intensity, and the solution process is very intuitionistic.

In the present paper, the equivalent non-linearization technique is utilized to evaluate the random response of the functionally graded material beam with viscoelastic damping under Gaussian white noise. Firstly, the dimensionless equation of the functionally graded material beam is derived, and then the viscoelastic force can be approximated to the quasi-linear damping and the quasi-linear stiffness by the generalized harmonic transformation. A reduced nonlinear system without viscoelastic term is established to approximate the original system with viscoelastic term and the stationary probability density of system displacement and velocity are analytically obtained through the equivalent non-linearization technique. The influences of the different parameters on the system response are discussed, especially the viscoelastic parameter $\beta$ and $\lambda$. The agreements between the analytical results and the results from Monte Carlo simulations validate the effectiveness of the proposed technique.

**Random Response Analysis**

The present paper is concerned with the functionally graded material beam with viscoelastic damping under random excitation modeled by Gaussian white noise, as shown in Figure 1.

The length of the FGM beam is $L$ and the thickness is $h$. Young’s modulus $E(z)$ and mass density $\rho(z)$ are assumed to change continuously along the thickness according to power distribution [11].

\[
E(z) = (E_m - E_c)(\frac{2z + h}{2h})^\nu + E_c,
\]

\[
\rho(z) = (\rho_m - \rho_c)(\frac{2z + h}{2h})^\nu + \rho_c,
\]

where the subscripts ‘c’ and ‘m’ denote the top surface of the beam (ceramic) and the
bottom surface (metal). $n$ is a constant characterizing the distributions of material properties.

By using the one-order Galerkin technique [12] and introducing the following dimensionless quantities, the motion equation can be written as

$$
\ddot{X} + 2\zeta \dot{X} + k_1X + k_3X^3 + Z = W(t) \quad (2)
$$

where $\zeta = \frac{cL}{2\sqrt{I_1(D_{11} - \frac{B_{11}^2}{A_{11}}})}$, $k_1 = \pi^4\frac{A_{11}^4h^2}{L^2}$, $k_3 = \frac{A_{11}^4h^2}{2(D_{11} - \frac{B_{11}^2}{A_{11}})h^2}$, $P_m = \frac{2^{3/2}L^2}{(D_{11} - \frac{B_{11}^2}{A_{11}})h\pi}$,

$$
\{A_{11}, B_{11}, D_{11}\} = \int_{-h/2}^{h/2} \frac{E(z)}{1 - v^2}\{1, z, z^2\}dz, \quad I_1 = \int_{-h/2}^{h/2} \rho(z)dz,
$$

and $Z = \int_0^t h(t - \tau)X(\tau)d\tau\cdot h(t) = \beta_1\exp(-t/\lambda_1)$ [13].

By using the generalized harmonic function, the viscoelastic integral term $Z$ in the system (2) can be expressed as the following algebraic form [14]

$$
\int_0^t h(t - \tau)X(\tau)d\tau = \frac{\beta_1\lambda_2^2}{1 + (\alpha(H)\lambda_1)^2} \ddot{X} + \frac{\beta_2\lambda_4}{1 + (\alpha(H)\lambda_1)^2} X + k_1X + k_3X^3 = W(t) \quad (3)
$$

Using the approximate relation formula (3), the viscoelastic damping in the functionally graded material beam can be approximated by the following equation of motion

$$
\ddot{X} + [2\zeta - \frac{\beta_1\lambda_2^2}{1 + (\alpha(H)\lambda_1)^2}] \ddot{X} + \frac{\beta_2\lambda_4}{1 + (\alpha(H)\lambda_1)^2} X + k_1X + k_3X^3 = W(t) \quad (4)
$$

In which

$$
\alpha(H) = 2\pi / \oint [2H - 2U(x)]^{-1/2}dx \quad U(x) = \left(k_1 + \sum_{i=1}^{m} \frac{\beta_i\lambda_i}{1 + (\alpha(H)\lambda_i)^2}\right)x^2 / 2 + k_3x^4 / 4
$$

By adopting the equivalent non-linearization technique, the stationary probability of system (4) is [15]

$$
p_x(x, \dot{x}) = N \exp\left[-\frac{1}{D_0} \int_0^H \left(2\zeta - \frac{\beta_1\lambda_2^2}{1 + (\alpha(H)\lambda_1)^2}\right)\ddot{x}\right] \left|_{H = \frac{x^2}{1 + (\alpha(H)\lambda_1)^2}} \right|_{x^2 = \frac{x^2}{1 + (\alpha(H)\lambda_1)^2}} \right|_{x^2 = \frac{x^2}{1 + (\alpha(H)\lambda_1)^2}}
$$

The correction term coefficients $a_1$ and $a_3$ in equation (5) can be defined as follows,

$$
\min_{a_1, a_3} E(e^2) \quad (6)
$$

in which $e = a_1X + a_3X^3 - \sum_{i=1}^{m} \frac{\beta_i\lambda_i}{1 + (\alpha(H)\lambda_i)^2}X$.

The stationary probability density function of the system (2) are

$$
p(x) = \int_{-\infty}^{\infty} p_x(x, \dot{x})dx, \quad \text{for} \quad p(x) = \int_{-\infty}^{\infty} p_x(x, \dot{x})dx, \quad (7)
$$
\[ p(\dot{x}) = \int_{-\infty}^{\infty} p_x(x, \dot{x}) \, dx. \] (8)

**Numerical Results and Discussions**

Some numerical calculations are carried out to validate the proposed analytical technique, and the analytical results (continuous lines) are compared with numerical simulations (dots) by Monte Carlo methods. Maxwell viscoelastic unit retains only one, system parameters are selected as \( E_c = 244.27 \text{ GPa}, \) \( \rho_c = 4429 \text{ kg/m}^3, \) \( v_c = 0.3, \) \( E_m = 122.56 \text{ GPa}, \) \( \rho_m = 3000 \text{ kg/m}^3, \) \( v_m = 0.3, \) \( h = 0.6, \) \( n = 1, \) \( c = 10000, \) \( \beta_1 = -0.1, \) \( \lambda_1 = 1.0 \) and \( D = 10000, \) unless otherwise mentioned.

First of all, the influence of two important viscoelastic parameters, the relaxation time \( \lambda_1 \) and the magnitude parameter \( \beta_1, \) on the response of the system was analyzed. Figures 2 and 3 relate the probability density function of the system displacement and velocity to the relaxation time \( \lambda_1 \) and magnitude parameter \( \beta_1. \) Figure 2 shows that the response of the system decreases with the increase of relaxation time \( \lambda_1. \) It can be clearly seen from Figure 2 that the analytical results (continuous lines) agree with the Monte Carlo simulation (MCS) results. As can be seen from Figure 3, when the magnitude parameter \( \beta_1 \) change from -0.3 to -0.1, the response of the system velocity increases, and when the absolute value of magnitude parameter \( \beta_1 \) becomes large, there are two peaks in the probability density function of the system displacement.

![Figure 2](image1.png)  
**Figure 2.** The influences of the relaxation time \( \lambda_1 \) on the viscoelastic system response.

![Figure 3](image2.png)  
**Figure 3.** The influences of magnitude parameter \( \beta_1 \) on the viscoelastic system response.
The influences of the functional gradient material gradient index $n$ and beam length ratio $L/h$ on the system response are given in the following. As shown in Figure 4, the increasing the material gradient index $n$ decreases the system response. Figure 5 shows that the system response increases while the beam length ratio $L/h$ is increased.

![Figure 4](image1.png)  
(a) system displacement  
(b) system velocity  
Figure 4. The influences of material gradient index $n$ on the viscoelastic system response

![Figure 5](image2.png)  
(a) system displacement  
(b) system velocity  
Figure 5. The influences of beam length ratio $L/h$ on the viscoelastic system response

**Conclusions**

In this paper, the effects of different parameters such as magnitude parameter and relaxation time, material gradient index, beam length ratio on the response of the system are discussed respectively. Reducing the relaxation time is the effective way to suppress the response of the system. The reduction of the beam length ratio and the increase in the material gradient index can decrease the system response.

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**References**


