Exponential Periodicity of Networks with Distributed Delays on Time Scales

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Keywords: Periodic Solution, Exponential Stability, Neural Networks, On Time Scales.

Abstract. This paper studies periodicity of networks with distributed delays on time scales. By using the contraction mapping theorem and inequalities on time scales, several sufficient conditions for the existence and exponential stability of periodic solution of such networks on time scales are derived. Moreover, we present an example to illustrate the feasibility of our results.

Introduction

In the past decades, neural networks have been extensively investigated owing to their promising applications in many areas such as associative memory, image processing, pattern recognition, signal processing, and combinatorial optimization.

Recently, people pay attention to the neural network models on time scales, and some of them have got some important results, such as [2,6,7]. In [6], authors considered the exponential stability, exponentially asymptotic stability, uniform and uniformly exponentially asymptotic stability for the trivial solution of set dynamic equations on time scales by using Lyapunov-like functions. In [7], existence and global exponential stability of anti-periodic solutions of networks with delays on time scales were concerned. In [2], authors studied existence and exponential stability of periodic solution for stochastic Hopfield neural networks on time scales. Especially, existence and global exponential stability of a periodic solution to interval general bidirectional associative memory in [8].

As we know, periodicity is one of the most important nature of the solution of neural network, and now people had got some important solutions.

In this paper, we consider a general class of neural networks with distributed delay on time scales:

\begin{equation}
\begin{gathered}
\dot{x}_i^\Delta(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} \int_{t-\tau}^{t} w_{ij}(t-s)f_j(x_j(s))\Delta s + I_i(t), t \in \mathbb{T}
\end{gathered}
\end{equation}

where $\mathbb{T}$ is a periodic time scale, $n$ is the number of neurons in layers, $x_i(t)$ corresponds to the state of the $i$th unit at time $t \in \mathbb{T}$, $f_j(x_j)$ are the input-output functions (the activation functions), $a_i(t)$ represents the rate with which the $i$th neuron will reset its potential to the resting state in isolation when they are disconnected from the network and the external inputs at time $t$, $b_{ij}(t)$ and $w_{ij}(t) : \mathbb{R} \to \mathbb{R}^+$ represent the weight coefficients of the neurons; $I_i(t)$ denotes the input of the $i$th neuron at time $t$; $i, j \in \mathbb{N}$, where $\mathbb{N} = \{1,2,\ldots,n\}$.

By using the contraction mapping theorem and inequality on time scales, we establish some new sufficient conditions on the existence and exponential stability of periodic solutions for (1). Obviously, our results are general new and can includes cases of $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$.

Preliminaries

In this section, we first introduce some basic definitions of dynamic equations on time scales. A
time scale is an arbitrary nonempty closed subset of the real numbers. In this paper, \( \mathbb{T} \) denotes an arbitrary time scale.

Definition 1: The forward and backward jump operators \( \sigma: \mathbb{T} \to \mathbb{T} \) and \( \rho: \mathbb{T} \to \mathbb{T} \) such that \( \sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \), \( \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \). And the graininess \( \mu: \mathbb{T} \to \mathbb{R}^+ \) is defined by \( \mu(t) := \sigma(t) - t \). Obviously, \( \mu(t) = 0 \) if \( \mathbb{T} = \mathbb{R} \), while \( \mu(t) = 1 \) if \( \mathbb{T} = \mathbb{Z} \).

A point \( t \in \mathbb{T} \) is said to be left (right)-dense if \( (\sigma(t), \mu(t)) = 0 \); A point \( t \in \mathbb{T} \) is said to be left (right)-scattered if \( (\rho(t), \mu(t)) > 0 \). If \( \mathbb{T} \) has a left-scattered maximum \( o \) then we let \( \mathbb{T}^e = \mathbb{T} / \{o\} \), otherwise \( \mathbb{T}^e = \mathbb{T} \).

Definition 2: We say that a time scale \( \mathbb{T} \) is periodic if there exists \( p > 0 \) such that if \( t \in \mathbb{T} \) then \( t \pm p \in \mathbb{T} \). For \( \mathbb{T} \neq \mathbb{R} \), the smallest positive \( p \) is called the period of the time scale.

Definition 3: A function \( f: \mathbb{T} \to \mathbb{R}^+ \) is called rd-continuous provided it is continuous at right dense points in \( \mathbb{T} \) and its left-sided limits exist at left-dense points in \( \mathbb{T} \). If \( f \) is continuous at each right-dense point and each left-dense point, then \( f \) is said to be a continuous function on \( \mathbb{T} \). The set of rd-continuous functions \( f: \mathbb{T} \to \mathbb{R}^+ \) will be denoted by \( \mathcal{R} = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) \).

Definition 4: A function \( f: \mathbb{T} \to \mathbb{R}^+ \) is called regressive provided \( 0 \neq 1 + \mu(t)f(t) \) for all \( t \in \mathbb{T} \), \( \Theta = \mathcal{R} = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) \).

Let \( p, q: \mathbb{T} \to \mathbb{R}^+ \) be two regressive functions, we define \( pq = \Theta \mu = \Theta \).

Definition 5: Let \( \mathcal{R} \in \mathbb{R}^+ \), the exponential function is defined by \( \exp(\Delta \zeta) \), \( \zeta, \tau, \sigma \in \mathbb{T} \), with the cylinder transformation \( \zeta_h = \begin{cases} \log(1+h \Z), & h \neq 0, \\ \zeta, & h = 0. \end{cases} \)

Lemma 1: Suppose that \( p \in \mathcal{R}^+ \), then
(i) \( e_p(t,s) > 0 \), for all \( t,s \in \mathbb{T} \),
(ii) if \( p(t) \leq q(t) \) for all \( t \geq s, t,s \in \mathbb{T} \), then \( e_p(t,s) \leq e_q(t,s) \), for all \( t \geq s \).

Lemma 2: If \( p \in \mathcal{R} \) and \( a,b,c \in \mathbb{T} \), then \( [e_p(c,t)]^a = -[e_p(c,t)]^a \) and \( \int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b) \)

For convenience, we denote \( [a,b]_\mathbb{T} = \{t \in [a,b] \cap \mathbb{T}\} \). For an \( \omega \)-periodic rd-continuous function \( f: \mathbb{T} \to \mathbb{R}^+ \), denote \( f = \sup_{r \in [0,\omega]} |f(t)| \), \( f = \inf_{r \in [0,\omega]} |f(t)| \). The initial condition of (1) is \( x_i(s) = \phi_i(s), -\infty < s \leq t_0, i \in N \) (2) where \( \phi_i \in \mathcal{C}_{rd} \).
Definition 6: The periodic solution \( x(t, t_0, \varphi) \) with initial value \( \varphi \) of (1) is said to be exponential stable, if there are positive constants \( \lambda \) with \( -\lambda \in \mathbb{R}^+ \) and \( M > 1 \) such that for any solution \( y(t, t_0, \varphi) \) with initial value \( \varphi \) of (1) satisfies

\[
\|x - y\| \leq M \|\varphi - \varphi_0\| e^{-\lambda(t - t_0)}, \quad \forall t \in [t_0, +\infty)_T
\]

Lemma 3: For any \( x \in \mathbb{R}^n_+ \) and \( p > 0 \),

\[
\left( \sum_{i=1}^{n} x_i \right)^p \leq n^{(p-1)n+1} \sum_{i=1}^{n} x_i^p
\]

Throughout this paper, we assume that:

(A1) \( a_i(t) > 0 \) with \( -a_i \in \mathbb{R}, b_j(t), w_j(t), I_i(t) \) are all periodic rd-continuous functions with period \( \omega \) for \( t \in \mathbb{T} \), where \( \mathbb{R} \) denotes the set of regressive functions on \( \mathbb{T} \), \( i, j \in \mathbb{N} \).

(A2) \( f_j \) are Lipschitz-continuous with Lipschitz constants \( L_j > 0 \), respectively, \( j \in \mathbb{N} \).

(A3) \( \int_0^{+\infty} w_j(s)\Delta s = \overline{w}_j < \infty \)

Main Results

For the existence of periodic solutions of (1), we introduce the following basic lemma.

Lemma 4: Let (A1)-(A3) hold. Then \( (x_1(t), x_2(t),..., x_n(t))^T \) is an \( \omega \)-periodic solution of (1) if and only if \( (x_1(t), x_2(t),..., x_n(t))^T \) is an \( \omega \)-periodic solution of the following:

\[
x_i(t) = \int_t^{t+\omega} G_i(t, s) \left[ \sum_{j=1}^{n} b_j(s) f_j(x_j(s)) + \sum_{j=1}^{n} \int_{-\infty}^{s} w_j(t-s) f_j(x_j(s))\Delta s + I_i(s) \right] \Delta s
\]

where

\[
G_i(t, s) = e_{-a_i(t+\omega, \sigma(t))}^{-a_i(t, \sigma(t))}, \quad \forall i \in \mathbb{N}
\]

Proof. Let \( (x_1(t), x_2(t),..., x_n(t))^T \) be an \( \omega \)-periodic solution of (1). Multiplying both sides of (1) by \( e_{-a_i(t, \sigma(t))} \), we have that

\[
[x_i(t)e_{-a_i(t_0, t)}] = \left[ \sum_{j=1}^{n} b_j(t) f_j(x_j(t)) + \sum_{j=1}^{n} \int_{-\infty}^{t} w_j(t-s) f_j(x_j(s))\Delta s + I_i(t) \right] e_{-a_i(t_0, \sigma(t))}
\]

where \( i \in \mathbb{N} \). It follows from \( x_i(t+\omega) = x_i(t) \) that

\[
x_i(t) = \int_t^{t+\omega} G_i(t, s) \left[ \sum_{j=1}^{n} b_j(s) f_j(x_j(s)) + \sum_{j=1}^{n} \int_{-\infty}^{s} w_j(s-v) f_j(x_j(v))\Delta s + I_i(s) \right] \Delta s
\]

which leads to (3). On the other hand, if \( (x_1(t), x_2(t),..., x_n(t))^T \) is an \( \omega \)-periodic solution of (3), one gets
\[
x_i^\Delta(t) = G_i(\sigma(t), t + \omega)\left[\sum_{j=1}^{n} b_{ij}(t + \omega) f_j(x_j(t + \omega)) + \sum_{j=1}^{n} \int_{-\infty}^{t} w_{ij}(t + \omega - v) f_j(x_j(v)) \Delta v + I_i(t + \omega)\right] \\
- G_i(\sigma(t), t)\left[\sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} \int_{-\infty}^{t} w_{ij}(t - v) f_j(x_j(v)) \Delta v + I_i(t)\right] - a_i(t) x_i(t) \\
= -a_i(t) x_i(t) + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} \int_{-\infty}^{t} w_{ij}(t - v) f_j(x_j(v)) \Delta v + I_i(t),
\]

Therefore, \((x_1(t), x_2(t), \ldots, x_n(t))^T\) is an \(\omega\)-periodic solution of (1). This completes the proof.

**Theorem 1:** Let (A1)-(A3) hold. Suppose that

\[
A(4) \Xi := \max_{1 \leq i \leq n} \left\{2\omega^2 \left(\sum_{j=1}^{n} b_{ij} L_j\right)^2 + \left(\sum_{j=1}^{n} w_{ij} L_j\right)^2\right\} B_i^2 < 1
\]

holds, where \(B_i = \max\{|G_i(t, s)| : t, s \in [0, \omega], t \leq s\}, i \in N\). Then (1) has a unique \(\omega\)-periodic solution.

**Proof:** By Lemma 4, the existence problem of \(\omega\)-periodic solutions of (1) is equivalent to that of \(\omega\)-periodic solutions of (3). Set \(\Gamma = \{x = (x_1, x_2, \ldots, x_n)^T \in C_{\text{ad}}(\mathbb{T}, \mathbb{R}^n) \mid x(t + \omega) = x(t)\}\). Then \(\Gamma\) is a Banach space with the norm

\[
\|x\| = \max_{1 \leq i \leq n} \sup_{t \in [0, \omega]} \left|x_i(t)\right|^2, i \in N
\]

Define an operator

\[
\Psi : \Gamma \to \Gamma, x = (x_1, x_2, \ldots, x_n)^T \mapsto \Psi x = ((\Psi x)_1, (\Psi x)_2, \ldots, (\Psi x)_n)^T
\]

where

\[
(\Psi x)_i(t) = \int_{t}^{t+\omega} G_i(t, s) \left[\sum_{j=1}^{n} b_{ij}(s) f_j(x_j(s)) + \sum_{j=1}^{n} \int_{-\infty}^{s} w_{ij}(s-v) f_j(x_j(v)) \Delta v + I_i(s)\right] \Delta s, i \in N
\]

It is clear that \(G_i(t + \omega, s + \omega) = G_i(t, s)\) for all \((t, s) \in \mathbb{T} \times \mathbb{T}\). Together with Lemma 4, we have

\[
(\Psi x)(t + \omega) = (\Psi x)(t)
\]

that is, \(\Psi x \in \Gamma\).

For any \(x = (x_1, x_2, \ldots, x_n)^T, x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \in \Gamma\), we have

\[
(\Psi x - \Psi x^*)_i(t) = \int_{t}^{t+\omega} G_i(t, s) \left[\sum_{j=1}^{n} b_{ij}(s) [f_j(x_j(s)) - f_j(x_j^*(s))] \Delta s \\
+ \sum_{j=1}^{n} \int_{-\infty}^{s} w_{ij}(s-v) [f_j(x_j(v)) - f_j(x_j^*(v))] \Delta v \right] \Delta s, i \in N
\]

By Lemma 3, for \(i \in N\), we have

\[
\left|(\Psi x - \Psi x^*)_i(t)\right| \leq 2\omega^2 \left(\sum_{j=1}^{n} b_{ij} L_j\right)^2 + \left(\sum_{j=1}^{n} w_{ij} L_j\right)^2 B_i^2 \|x - x^*\|
\]

i.e.,

\[
\left\|(\Psi x - \Psi x^*)_i(t)\right\| \leq \Xi \|x - x^*\|
\]

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By (A4), we have that $\Psi$ is a contraction mapping. Hence, $\Psi$ has a unique fixed point, which implies that (1) has a unique $\omega$-periodic solution. This completes the proof of Theorem 1.

Theorem 2: Let (A1) - (A4) hold. If there exists $\eta_i \in C_{\omega}$ with $0 < 1 + \mu(t)\eta_i(t) < 1$ and

$$\lim_{t \to +\infty} \int_{t_0}^{t} \xi_{\mu(t)}(\eta_i(\tau))\Delta \tau = -\infty, \forall i \in N$$

such that

$$\Lambda = \inf_{\eta \in [0, \infty)} \max \{-(a_i + \eta_i(t))e_{\eta_i}(t, t_0) + \sum_{j=1}^{n} (b_{ij} + w_{ij})L_{ij}e_{\eta_i}(t, t_0) \} < 0$$

then the periodic solution of (1) is exponentially stable.

Proof. By Theorem 1, (1) has a unique $\omega$-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))^T$ with initial condition $\phi^* = (\phi_1^*(s), \phi_2^*(s), ..., \phi_n^*(s))^T$. Suppose that $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ is an arbitrary solution of (1) with initial condition $\phi(s) = (\phi_1(s), \phi_2(s), ..., \phi_n(s))^T$. Denote $y(t) = (y_1(t), y_2(t), ..., y_n(t))^T$, where $y_i(t) = |x_i(t) - x_i^*(t)|, i \in N$. Then we have

$$y_i^A(t) = -a_i y_i(t) + \sum_{j=1}^{n} b_{ij} \int_{t}^{t-s} w_{ij}(t-s)[f_j(x_s(t)) - f_j(x_j^*(t))]\Delta s$$

$$\leq -a_i y_i(t) + \sum_{j=1}^{n} b_{ij} L_{ij} y_j(t) + \sum_{j=1}^{n} \int_{t}^{t-s} w_{ij}(t-s) L_{ij} y_j(s)\Delta s,$$

where $i \in N$. Let

$$y(t_0) = \max_{i \in \mathbb{N}} \sup_{s \in [0, t_0]} |\phi_i(s) - \phi_i^*(s)|$$

For $t \in \mathbb{T}_\omega$, denote

$$z_i(t) = \begin{cases} y_i(t) e_{\xi_i}(t, t_0), \quad t \in (t_0, +\infty) \mathbb{T}, \\ y_i(t), \quad t \in (-\infty, t_0) \mathbb{T}, \end{cases}$$

where $\xi \in C_{\omega}$ and $0 < 1 + \mu(t)\xi(t) < 1$. Let $\phi(t) = z_i(t) - y(t_0)e_{\xi i}(t, t_0)$. Due to $\phi(t) \leq 0$ for $t \in (-\infty, t_0) \mathbb{T}$, we will have $\phi(t) \leq 0$, for $t \in [t_0, +\infty) \mathbb{T}$. Otherwise, there exist a subset $N^i \subseteq N$ and a first time $t_i \geq t_0$ such that

$$\begin{cases} \phi_i(t_0) = 0 \text{ and } \phi_i^A(t_1) > 0, \\ \text{or } \phi_i(\rho(t_1)) \leq 0 \text{ and } \phi_i(t_i) > 0, \forall k \in N^i \\ \phi_i(t) \leq 0, \quad t \in (-\infty, t_i) \mathbb{T}, \forall i \in N \setminus N^i. \end{cases}$$

From (6), one has
\[ D^\nu \phi_k^*(\rho(t)) = D^\nu z_k^*(\rho(t)) - y(t_0)e_{\eta_k}(\rho(t), t_0) \]
\[ = y_k^*(\rho(t_0))e_{\eta_k}(\rho(t), t_0) + \sum_{j=1}^{n} b_{ij}L \eta_j(\rho(t), t_0) + \sum_{j=1}^{n} w_{ij}(\rho(t), t_0) \Delta s(1 + \mu(\rho(t))) \]
\[ \leq ( -a_{ij}y_k(\rho(t)) + \sum_{j=1}^{n} b_{ij}L \eta_j(\rho(t), t_0) + \sum_{j=1}^{n} w_{ij}(\rho(t), t_0) \Delta s(1 + \mu(\rho(t))) ) \]
\[ \times (1 + \mu(\rho(t))) + \varepsilon_j(\rho(t))e_{\eta_j}(\rho(t), t_0) - \eta_j(\rho(t)) \mu(\rho(t)) \]
\[ \times \eta_j(\rho(t))e_{\eta_j}(\rho(t), t_0) \]
\[ \leq ( -a_{ij} + \eta_j(\rho(t)) )e_{\eta_j}(\rho(t), t_0) + \sum_{j=1}^{n} (b_{ij} + w_{ij}) \Delta s \]
\[ \times k y(t_0)(1 + \varepsilon_j(\rho(t))\mu(\rho(t)))e_{\eta_j}(\rho(t), t_0) < 0 \]
which contradicts to (7). Hence, for \( \forall t \geq t_0 \) and \( \forall i \in N \),
\[ y_i(t) \leq y(t_0)e_{\eta_i}(t, t_0) = y(t_0)\exp(\int_{t_0}^{t} \xi(\tau)(\eta_i(\tau)\Delta \tau) \rightarrow 0 \text{ as } t \rightarrow \infty \]

I.e.,
\[ \| x - \xi \| \leq e_{\eta_i}(t, t_0)\| \varphi - \varphi^* \| \]
that is, the periodic solution \( \dot{x}(t) \) of (1) is exponentially stable. This completes the proof of Theorem 2.

**Examples**

In this section, we will give an example to illustrate Theorem 1 and Theorem 2.

Example 1: Let \( n = 2 \). Consider the following neural network on the periodic time scale \( \mathbb{T} = \bigcup_{k=2}[2k\pi, (2k+1)\pi] \) with the period \( \omega = 2\pi \): \[ x_i^*(t) = -a_{ii}x_i(t) + \sum_{j=1}^{2} b_{ij}f_j(x_j(t)) + \sum_{j=1}^{2} w_{ij}(t-s)f_j(x_j(s))\Delta s + I_i(t) \]
where \( t \in \mathbb{T}, i = \{1, 2\} \) and the coefficients are as follows:
We have that \( L_1 = 0.1 \), \( L_2 = 0.2 \), \( \omega = 2\pi \).

\[
\mu(t) = \begin{cases} 
0, & t \in \bigcup_{k \in \mathbb{Z}} [2k\pi, (2k+1)\pi), \\
\pi, & t \in \{(2k+1)\pi\},
\end{cases}
\]

\[
a(t) = \begin{bmatrix} 1.05 + 0.05 \cos t \\
1.03 + 0.02 \sin 2t \end{bmatrix},
b(t) = \begin{bmatrix} 0.4 \cos t & 0.2 \sin t \\
0.3 \sin t & 0.1 \cos t \end{bmatrix},
W(t-s) = \begin{bmatrix} \frac{2}{3\pi [1+(t-s)\pi]} & 0 \\
0 & \frac{1}{2e^{(t-s)}} \end{bmatrix}
\]

\[
f_1(u) = 0.1 \cos u, \ f_2(u) = 0.2 \sin u, \ I_1(t) = I_2(t) = 0.7 \cos t, \eta_i(t) = \frac{1}{7}, \ i = 1,2.
\]

\[
\int_{-\infty}^{\infty} W(t-s) \Delta s = \begin{bmatrix} \frac{1}{7} & 0 \\
0 & \frac{1}{7} \end{bmatrix},
\]

\[
Z_1 = 2\omega^2 \left( \sum_{j=1}^{n} \bar{b}_1 \bar{L}_j \right)^2 + \left( \sum_{j=1}^{n} \bar{w}_1 \bar{L}_j \right)^2 B_1^2 \leq 8\pi^2 (0.04 + 0.04)^2 + (\frac{1}{7} \times 0.1)^2 \times 1 \approx 0.593
\]

\[
Z_2 = 2\omega^2 \left( \sum_{j=1}^{n} \bar{b}_2 \bar{L}_j \right)^2 + \left( \sum_{j=1}^{n} \bar{w}_2 \bar{L}_j \right)^2 B_2^2 \leq 8\pi^2 (0.03 + 0.02)^2 + (\frac{1}{7} \times 0.2)^2 \times 1 = 0.395
\]

\[
\Lambda_1 = (\frac{1}{7} - 0.92 + \frac{1}{30}) e_{-\gamma}(t,t_0), \quad \Lambda_2 = (\frac{1}{7} - 0.91) e_{-\gamma}(t,t_0).
\]

Hence, \( \Xi = \max\{Z_1, Z_2\} < 1 \) and \( \Lambda = \max\{\Lambda_1, \Lambda_2\} < 0 \). It follows from Theorems 1 and 2 that (8) has an exponentially \( 2\pi \)-periodic solution.

**Summary**

By using contraction mapping and inequalities on time scales, we gave the sufficient conditions for the considered neural networks have been established on time scales. An example with simulations was also given to show the effectiveness of the obtained result. We would like to point out that it is possible to generalize our main results to more complex neural networks, such as mixed time-varying delayed neural networks with discontinuous activations [4], mixed time-varying delayed neural networks [5], cellular neural networks [3].

**Acknowledgement**

This research was supported by the National Natural Science Foundation of China under Grants 61573005 and 11361010, the Foundation for Young Professors of Jimei University, the Excellent Youth Foundation of Fujian Province under Grant 2012J06001, NCETFJ under Grant JA11144 and the Foundation of Fujian Higher Education under Grants JA11154.

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