Exact Analysis of Weak Formulation Study for Thick Laminated Shell

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ABSTRACT

Weak formulations of mixed state equations including boundary conditions of open laminated cylindrical shell are presented in a cylindrical coordinate system. The mixed state equation of an open cylindrical shell is established. The mixed equation is changed into Hamilton canonical equation. For applying the transfer matrix method and taking the advantage of Hamiltonian matrix in the calculation, an unified approach and three-dimensional solutions are obtained for open thick laminated cylindrical shell with relative edges simply supported and the relative edge clamped supported subjected to mechanical loading. Numerical results are given to compare with those of FEM calculated using SAP5. The principle and method suggested here have clear physical concepts. The equations and boundary conditions proposed in this paper are weakened, the method of this paper would be easy to popularize in dynamics analysis of elasticity.

INTRODUCTION

The rapidly growing applications of fibre-reinforced composites in high-performance aerospace vehicles have led to intensive study of the behavior of laminated composite structures under various conditions. For plate-shell composite structures analysis, this has already received wide spread attention. Since the current theories of plates and shells are established on some hypotheses, only partial fundamental equations can be satisfied and some of the elastic constants can not be taken into account. Because of this, the errors will increase as the thickness of plate or shell increase, and the stresses at interface can not be exactly calculated. In addition, all of the existing theories are invalid for quite thick plates and shells.

With no initial assumptions regarding stress and deformation models [1-4], and using three-dimensional elasticity the free vibration problems of homogeneous isotropic, orthotropic, and laminated thick cylindrical shells were solved. In these papers, the thick shells were divided into \(N\) fictitious subcylinders in order to simplify the variable coefficient differential equations into a set of simpler ones that was solved by using a method of successive approximations. The frequencies with
desired accuracy were obtained, by increasing the value of \( N \). A similar approach was suggested by Bhimaraddi [5] to analyze the free vibration of doubly curved shallow shells. But all of the papers had to deal with the many unknowns that appear at the real or fictitious interfaces. Based on the three-dimensional theory of elasticity, the buckling and vibration of isotropic, orthotropic, and laminated cylindrical shells has also been studied by Kardomateas [6-7], and by Ye and Soldatos [8-9]. Khdeir [10] investigated thermal deformations and stresses in cross-ply laminated circular cylindrical shells by means of state space approach. Ding and Tang [11-14] developed the method of state space, and gave the exact solution for axisymmetric vibration and buckling of laminated cylindrical shells having simply supported edge boundary and clamped edges, respectively. Exact thermoelastic solution for an axisymmetric problem of thick closed laminated shells has also been studied by Ding and Tang [15]. Ding [16] obtained three-dimensional solutions of thick closed laminated shell by means of weak formulation study. Hedayati H. [17] invests semi-analytical 3D elasticity solutions for open cylindrical shells under various boundary conditions.

To the author’s knowledge, the three-dimensional analysis for the quite thick open laminated cylindrical shell with relative edges simply supported and the relative edge clamped supported is so difficult that few references have been found. In this paper, however, based upon weak formulation of mixed state equations including boundary conditions, the mixed variational formulation of open laminated cylindrical shell is established. A three-dimensional solution is expressed for the thick open laminated shell of arbitrary thickness having various boundary conditions subjected to mechanical loading. Numerical results are obtained and compare with those of FEM calculated using SAP5.

**WEAK FORMULATION OF MIXED STATE EQUATIONS**

A open cylindrical shell is investigated. The principal elastic directions of the shell coincide with the coordinate axes. Let \( u \), \( v \) and \( w \) be the displacement in the \( x-\theta-\) and \( r-\) directions, respectively. Also, let \( V, S \) and \( f_i \) be volume, boundary surface and body forces, respectively. \( p_x, p_\theta \) and \( p_r \) are the surface forces in the \( x-\theta-\) and \( r-\) directions, respectively.

The equilibrium equation in the \( x-\) direction for \( \delta V \) can be written as

\[
\iint_{SV} p_x ds + \iint_{SV} f_x dV = 0
\]

By means of Green-formulation we can obtain

\[
\iiint_V \left( \frac{\partial \tau_{x\theta}}{\partial x} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rx}}{\partial r} + \frac{\tau_{rx}}{r} + f_x \right) dV + \iint_{SV} \left( \bar{p_x} - p_x \right) ds = 0 \quad (1a)
\]

the same as above, we have

\[
\iiint_V \left( \frac{\partial \tau_{o\theta}}{\partial x} + \frac{1}{r} \frac{\partial \tau_{o\theta}}{\partial \theta} + \frac{\partial \tau_{ro}}{\partial r} + \frac{2 r \tau_{ro}}{r} + f_\theta \right) dV + \iint_{SV} \left( \bar{p_\theta} - p_\theta \right) ds = 0 \quad (1b)
\]
\[
\iint_V \left( \frac{\partial \tau_{rs}}{\partial x} + \frac{1}{r} \frac{\partial \tau_{rt}}{\partial \theta} + \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_{t\theta}}{r} + f_r \right) dV + \iint_{S_r} (p_r - p_r) ds = 0
\]  
(1c)

\[
\iint_V \left( \frac{\partial u}{\partial x} - \varepsilon_{rs} \right) dV + \iint_{S_u} (u - u) n_r ds = 0
\]  
(2a)

\[
\iint_V \left( \frac{\partial v}{\partial \theta} - \varepsilon_{t\theta} \right) dV + \iint_{S_v} (v - v) n_\theta ds = 0
\]  
(2b)

\[
\iint_V \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} - \gamma_{r\theta} \right) dV + \iint_{S_u} (u - u) n_\theta ds + \iint_{S_v} (v - v) n_\theta ds = 0
\]  
(2f)

in which the usual index notation is used. The stress-strain relations of cylindrical orthotropy is

\[
\{\sigma\} = [C]\{\varepsilon\}
\]  
(3)

Where
\[
\{\sigma\} = \begin{bmatrix} \sigma_s & \sigma_{t\theta} & \sigma_r & \tau_{r\theta} & \tau_{rx} & \tau_{x\theta} \end{bmatrix}^T
\]
\[
\{\varepsilon\} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{r} \frac{\partial v}{\partial \theta} & w & \frac{\partial w}{\partial r} & \frac{\partial \varepsilon_{rs}}{\partial \theta} & \frac{\partial \varepsilon_{t\theta}}{\partial r} & \frac{\partial \varepsilon_{t\theta}}{\partial \theta} & \frac{\partial \varepsilon_{t\theta}}{\partial x} & \frac{\partial \varepsilon_{t\theta}}{\partial \varepsilon_{t\theta}} \end{bmatrix}^T
\]

the matrix [C] is the elastic stiffness matrix.

Substituting Eq. (3) into Eqs. (2a)-(2f), then integrating by the weight function, i.e., multiply Eqs. (1a)-(1c) by \(\delta u\), \(\delta v\), \(\delta w\) and multiply Eqs. (2a)-(2f) by \(\delta \sigma_s\), \(\ldots\), \(\delta \tau_{r\theta}\), respectively.

One denotes \(q = (uvw)^T\), \(p = (\tau_{rx} \tau_{r\theta} \sigma_r)^T\), \(p_1 = (\sigma_s \sigma_{t\theta} \tau_{x\theta})^T\), the following relations can be obtained

\[
\begin{align*}
\iint_V \left( \frac{\partial \phi}{\partial r} + \frac{\partial H}{\partial q} \right) dV + \Gamma_1 &= 0 \\
\iint_V \left( \frac{\partial q}{\partial r} - \frac{\partial H}{\partial \phi} \right) dV + \Gamma_2 &= 0 \\
\iint_V \left( Dq - \frac{\partial H}{\partial p_1} \right) dV + \Gamma_3 &= 0
\end{align*}
\]  
(4)

Eq. (4) can also be written in a simplified form

\[
\frac{\partial \phi}{\partial r} = \frac{\partial H}{\partial \phi}, \quad \frac{\partial q}{\partial r} = -\frac{\partial H}{\partial q}
\]  
(5)

This is classical Hamilton canonical equation [11,12].
Selecting $F = (p, q)^T$ constitute state vector, Eq.(5) weak formulation of mixed state equation including boundary conditions of open cylindrical shell can be obtained

$$\iiint_V \frac{\partial}{\partial r} F \cdot dV = \iiint_V \frac{\partial}{\partial r} (HF + D_1 p_1) dV + \int_S \frac{\partial}{\partial S_1} ds$$

$$\iiint_V \frac{\partial}{\partial \phi} \cdot (D_2 F + B p_1) dV + \int_S \frac{\partial}{\partial S_2} ds = 0$$

**THE WEAK FORMULATION SOLUTION**

An orthotropic thick open cylindrical shell is investigated, in which the boundaries $(\theta = 0, \phi)$ are simply supported and the two other edges $(x = 0, l)$ are clamped. The length of the shell is $l$, the radii of outer and inner surfaces are $a$ and $b$, respectively. We introduce

$$u = u_x + \left(1 - \frac{x}{l}\right) u^{(0)} r + \frac{x}{l} u^{(l)}$$

in which $u^{(0)}$ and $u^{(l)}$ are unknown coefficients at $x = 0, l$, respectively.

Expand the quantities into the series system. Then we obtain the mixed state Hamilton equation for open cylindrical shell for each combination of $m$ and $n$

$$\frac{d}{dr} F(r) = M(r) F(r) + S(r)$$

where

$$F(r) = \begin{bmatrix} ru_{mn}(r) & rv_{mn}(r) & rw_{mn}(r) & \tau_{r, mn}(r) & \tau_{r, \theta, mn}(r) & \sigma_{r, mn}(r) \end{bmatrix}^T,$$

$$M(r) = \begin{bmatrix} A^T & D \\ E & -A \end{bmatrix}$$

In order to solve Eq. (9), thick shell should be divided into some thin plies. If we find, from calculation, that the needful effective digits hardly change, it can be said that the results obtained with certain thin plies are exact within the prescribed accuracy limits. For the first ply, the solution of Eq. (10) is

$$F(r) = G(r - a) F(a) + C(r - a)$$

in which

$$G(r - a) = e \times M[ \cdot r (-a) C(r - a) = \int_r e \times M \cdot r (x - a) d$$

Because $M$ is Hamiltonian matrix, $G(r - a), C(r - a)$ can be calculated by means of symplectic algorithm [18].

Let $r$ be equal to inner radius $r_i$ of first thin ply in Eq. (11), and consider $r_i - a = -h_1$, yields

$$F(r_i) = G(-h_1) F(a) + C(-h_1)$$

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On the analogy of this, the mechanical quantities of the interior and outer surfaces for the entire laminated shell can be linked together to be of the form

\[ F(b) = \Pi F(a) + \Pi \]

where

\[ \Pi = G(-h_k)G(-h_{k-1})G(-h_{k-2})\cdots G(-h_2)G(-h_1) , \]

\[ \Pi = G(-h_k)\sum_{j=1}^{k-1} \prod_{i=k-j}^{j} G(-h_i)C(-h_j) + C(-h_k) \]

\[ F(a) \text{ and } F(b) \text{ in Eq. (14) are the mechanical quantities for the interior and outer surfaces of the laminated shell, respectively. Usually, the loads acting on the interior and outer surfaces of a shell are given a priori. Actually, Eq. (15) is a matrix equation for six displacements of the interior and outer surfaces of a shell.} \]

Selecting of the fourth, fifth and sixth rows of matrix Eq. (14) gives

\[
\begin{align*}
\begin{pmatrix}
au_{mn}(a) \\
au_{mn}(b) \\
aw_{mn}(a) \\
aw_{mn}(b)
\end{pmatrix} &= 
\begin{pmatrix}
\Pi_{41} & \Pi_{42} & \Pi_{43} \\
\Pi_{51} & \Pi_{52} & \Pi_{53} \\
\Pi_{61} & \Pi_{62} & \Pi_{63}
\end{pmatrix}^{-1}
\begin{pmatrix}
\tau_{r,mn}(b) \\
\tau_{r,mn}(a) \\
\sigma_{r,mn}(b) \\
\sigma_{r,mn}(a)
\end{pmatrix} - 
\begin{pmatrix}
\Pi_{44} & \Pi_{45} & \Pi_{46} \\
\Pi_{54} & \Pi_{55} & \Pi_{56} \\
\Pi_{64} & \Pi_{65} & \Pi_{66}
\end{pmatrix}
\begin{pmatrix}
\tau_{r,mn}(a) \\
\tau_{r,mn}(a) \\
\sigma_{r,mn}(a)
\end{pmatrix} - 
\begin{pmatrix}
\Pi_{44} \\
\Pi_{54} \\
\Pi_{64}
\end{pmatrix} \\
\begin{pmatrix}
\Pi_{45} \\
\Pi_{55} \\
\Pi_{65}
\end{pmatrix} \\
\begin{pmatrix}
\Pi_{46} \\
\Pi_{56} \\
\Pi_{66}
\end{pmatrix}
\end{align*}
\]

Using boundary conditions the \( \Pi_3, \Pi_4, \Pi_5 \) can be determined.

**NUMERICAL EXAMPLE**

Consider a three-plied open laminated shell which is loaded by uniform normal pressure \( q \) on the top surface. The materials for the first and third layers are identical, where \( C_{11}^{(i)} \) and \( C_{11}^{(2)} \) denote \( C_{11} \) of the materials corresponding to the first and second layer, respectively. The laminated shell has the following geometry parameters \( h_1 = h_3 = 0.1h \), \( h_2 = 0.8h \), \( l = s = 2\pi R \). where \( l \) the length of the shell, \( s \) = the arc length of middle surface and \( R_0 \) is the radius of middle surface.

Some numerical results are obtained and shown in Fig. 1 the variations of the displacement and stresses, through the thickness at \( x = l/2 \), \( \theta = \pi/2 \) of thick laminated shells with different ratios \( h/R_0 = 0.8 \) and 1.0, respectively. The results for three-dimensional finite element method (FEM) using SAP5 (Structural Analysis Program 5) are also given in Fig.1. Because of the symmetry, 128 three-dimensional isoparametric elements (for 1/2 shell) with 20 nodes are employed in calculation. The average errors of present results and those of SAP5 are 3.17%. Where \( w^* = wC_{11}^{(2)}/(qh) \), \( u^* = uC_{11}^{(2)}/(qh) \) and \( \sigma_{r}^* = \sigma_{r}/q, \sigma_{\theta}^* = \sigma_{\theta}/q \).
CONCLUSION

Weak formulation of mixed state equation including boundary conditions are presented. The thick open laminated shell with relative edges simply supported and the relative edge clamped supported subjected to mechanical loading is investigated. The principle and method suggested here have clear physical concepts and overcome the contradictions and limitations that arise from incompatibility among the fundamental equations in various theories of plates and shells. Numerical results denote that the methods of dividing the layer into several thin plies has the characteristics of fast convergence rate, satisfactory precision, and controlled error. The present study satisfies the continuity conditions of stresses and displacements at the interfaces. Numerical results show that this method is effective. The solutions and results given here may serve a useful purpose in providing a check upon other solution procedures.

ACKNOWLEDGMENTS

This project 11472005 supported by National Natural Science Foundation of China and Anhui Provincial Science and Technology Research Project Funding through grant No.1501041133.

REFERENCES


